GOVERNMENT DEGREE COLLEGE , RAVULAPALEM

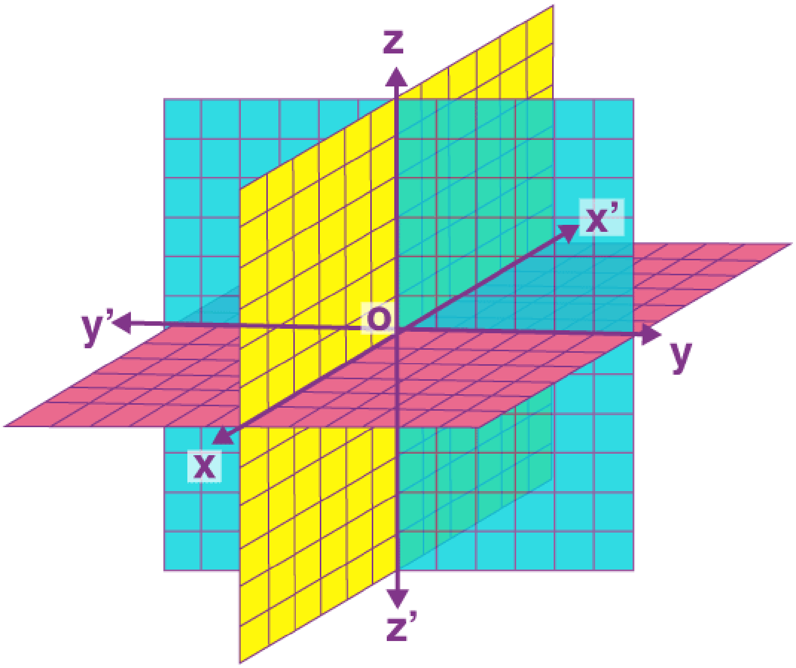
(An Outcome based educational institution since 1981)

Affiliated to Aadi kavi Nannaya University

First B.SC Second Semester 2022-23

Solid Geometry (3D-Geometry)

Unit I. Coordinates and the plane



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Solid Geometry (3D-Geometry)

Coordinate

Formulae:

1. In the three-dimensional space equations of X-axis is y = 0, z = 0.

Y-axis is x = 0, z = 0.

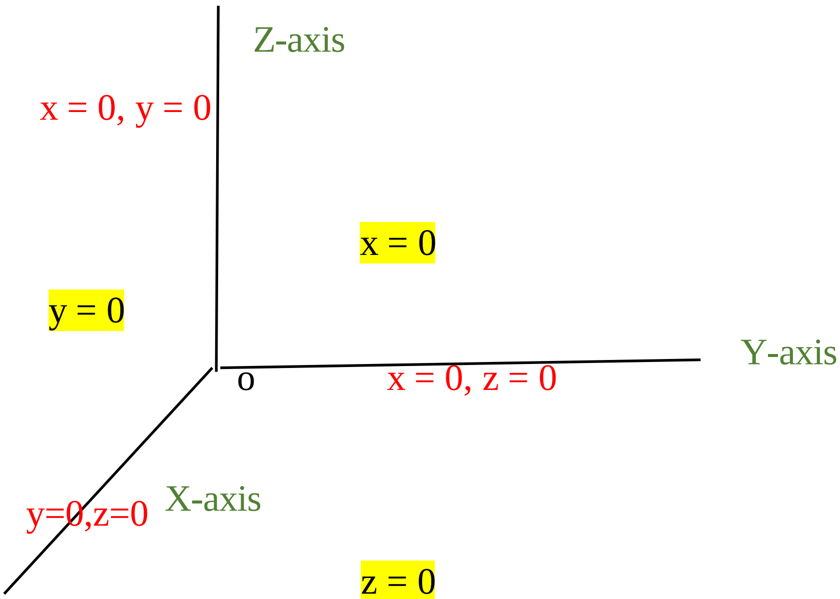
Z- axis is x = 0, y = 0.

2. In the three-dimensional space equations of YZ – Plane is x = 0

XZ – Plane is y = 0,

XY – Plane is z = 0

3. Any point in three-dimensional space contains x, y, z coordinates.



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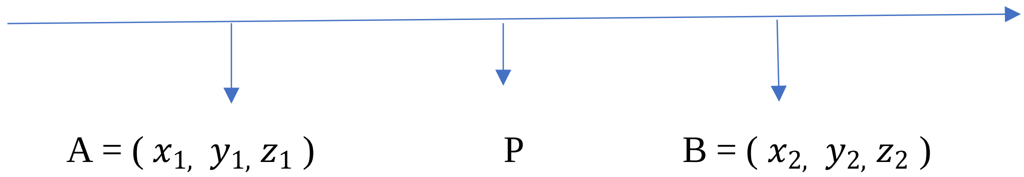
4. Let A = (x_1, y_1, z_1) B = (x_2, y_2, z_2) are any two points then the distance between A and B is

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

5. Let P = (x, y, z) then OP = $\sqrt{x^2 + y^2 + z^2}$ where P = (0, 0, 0)

6. Let A = (x_1, y_1, z_1) B = (x_2, y_2, z_2) are any two points then the midpoint of A, B is

$$P = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2} \right)$$



7. A-point P when it divides A = (x_1, y_1, z_1) B = (x_2, y_2, z_2) in the ratio m: n internally is

$$P = \left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}, \frac{mz_2 + nz_1}{m+n} \right)$$

and the external point

$$P = \left(\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n}, \frac{mz_2 - nz_1}{m-n} \right)$$

8. The centroid the Triangle ABC where A = (x_1, y_1, z_1) B = (x_2, y_2, z_2) and C = (x_3, y_3, z_3) is

$$G = \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3} \right)$$

9. The centroid the Tetrahedron ABCD where

A = (x_1, y_1, z_1) B = (x_2, y_2, z_2) C = (x_3, y_3, z_3) D = (x_4, y_4, z_4) is

$$G = \left(\frac{x_1 + x_2 + x_3 + x_4,}{4}, \frac{y_1 + y_2 + y_3 + y_4,}{4} + , \frac{z_1 + z_2 + z_3 + z_4,}{4} \right)$$

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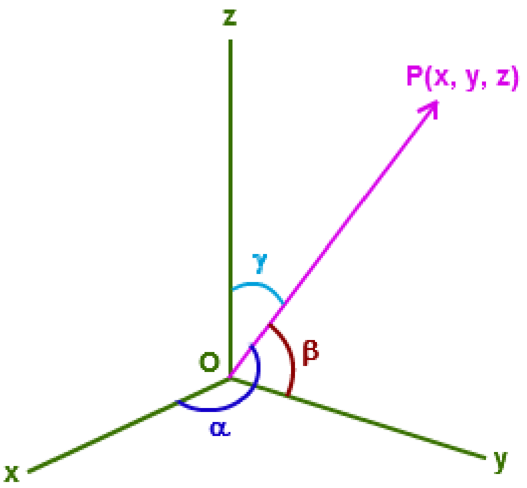

10. Direction cosines and Direction ratios:

Let A = (x_1, y_1, z_1) B = (x_2, y_2, z_2) are any two points then the direction ratios of the line AB are $x_2 - x_1, y_2 - y_1, z_2 - z_1$.

11.A line making angles α, β, γ with the coordinate axes X, Y and Z then the Cosine angles of α, β, γ that is $\cos\alpha, \cos\beta, \cos\gamma$ called Direction cosines of the line and they are denoted by l, m, n.

Note: If l, m, n are direction cosines of a line L then $l^2 + m^2 + n^2 = 1$.

12. The Dices of a line AB where A = (x_1, y_1, z_1) B = (x_2, y_2, z_2) are

$$\frac{x_2 - x_1}{AB} = \frac{y_2 - y_1}{AB} = \frac{z_2 - z_1}{AB}$$

Where AB = $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

13. If (l_1, m_1, n_1) (l_2, m_2, n_2) are dices of the lines L_1, L_2 then the angle between then $\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2$

14. If $l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$ *then the lines are perpendicular*

15. If $\frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}$ then the lines are parallel

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16.The projection of the line A = ($x_{1,}$ $y_{1,}$ z_1) B = ($x_{2,}$ $y_{2,}$ z_2) on the line

L whose Dices are l, m, n is l $(x_2 - x_1) + m(y_2 - y_1) + n (z_2 - z_1)$

17.The dices of X-axis (1,0,0)

Y-axis (0,1,0)

Z-axis (0,0,1),

18.The dices of YZ-plane (1,0,0)

ZX-plane (0,1,0)

XY-plane (0,0,1)

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2.The Plane

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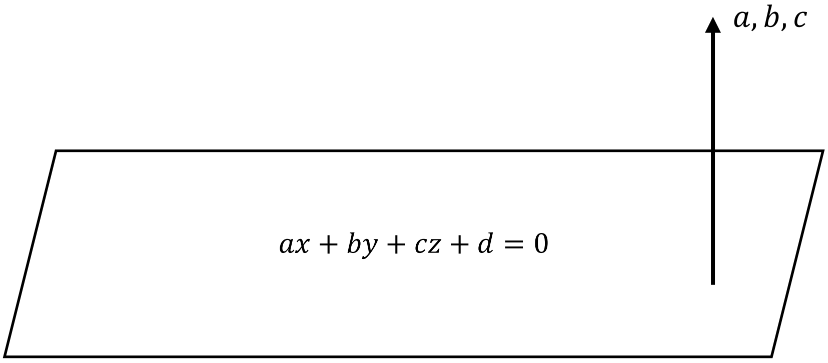
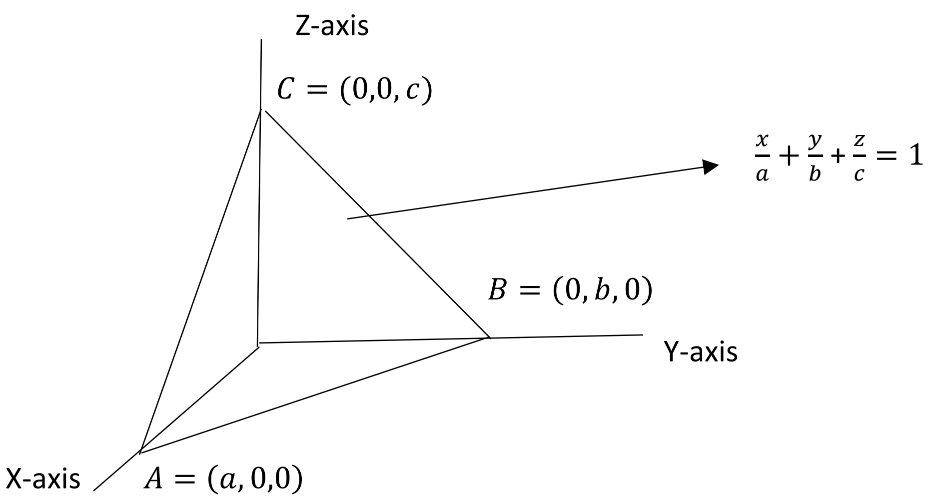
Definition: A surface is called a plane surface or plane if all the points of a straight line joining any two points on the surface lie on it.

1) The general equation of a plane $ax + by + cz + d = 0$ where a, b, c are direction ratios of the normal line to the plane and $a^2 + b^2 + c^2 \neq 0$ and if d = 0 then it is passing through origin.

2) The normal form of plane is $lx + my + nz = p$ where l, m, n are direction cosines of the normal line to the plane that is $l^2 + m^2 + n^2 = 1$.



3)The intercepting form of the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ where $A = (a, 0,0), B = (0, b, 0)$ and $C = (0,0, c)$ are X, Y *and* Z *intercepts* of the plane.



4)The normal form of the plane $ax + by + cz + d = 0$ is

$$\frac{a}{\sqrt{a^2+b^2+c^2}}\, x + \frac{b}{\sqrt{a^2+b^2+c^2}}\, y + \frac{c}{\sqrt{a^2+b^2+c^2}}\, z = \frac{-d}{\sqrt{a^2+b^2+c^2}}$$

5)The perpendicular distance from the point P ($x_{1,}y_1,z_1$) to the plane

$ax + by + cz + d = 0$ is $\frac{|\, ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}$

6)The perpendicular distance from the origin to the plane $ax + by + cz + d = 0$ is

$$\frac{|\, d|}{\sqrt{a^2+b^2+c^2}}$$

7)Equation of a plane passing through P ($x_{1,}y_1,z_1$) and whose direction ratios of the normal line are a, b, c is

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0.$$

8)Angle between the planes $a_1x + b_1y + c_1$z $+d_1$=0 and $a_2x + b_2y + c_2$z $+d_2$=0 is

$$\cos\theta = \frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1{}^2+b_1{}^2+c_1{}^2}\,\sqrt{a_2{}^2+b_2{}^2+c_2{}^2}}$$

(i). If $a_1a_2 + b_1b_2 + c_1c_2 = 0$ then the planes are perpendicular.

(ii). If $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$ then the planes are parallel.

9)The parallel plane to a plane $ax + by + cz + d = 0$ is in the form $ax + by + cz + k = 0$.

10)The distance between the parallel planes

$ax + by + cz + d_1 = 0$ & $ax + by + cz + d_2 = 0$ is $\frac{|\ d_1 - d_2\ |}{\sqrt{a^2 + b^2 + c^2}}$

11) i). The direction cosines of the XY-Plane

= direction cosines of the Z-axis =0, 0, 1

ii). The direction cosines of the YZ-Plane

= direction cosines of the X-axis =1, 0, 0

iii). The direction cosines of the ZX-Plane

= direction cosines of the Y-axis =0, 1, 0

Problems:

1.Reduece the equation $2x - 6y + 3z - 21 = 0$ of a plane into normal form.

Solution: Equation of the given plane $2x - 6y + 3z - 21 = 0$

And it's normal form

$$\frac{a}{\sqrt{a^2+b^2+c^2}}x + \frac{b}{\sqrt{a^2+b^2+c^2}}y + \frac{c}{\sqrt{a^2+b^2+c^2}}z = \frac{-d}{\sqrt{a^2+b^2+c^2}}$$

$$\sqrt{a^2 + b^2 + c^2} = \sqrt{2^2 + (-6)^2 + 3^2} = \sqrt{49} = 7$$

$$\therefore \frac{2x-6y+3z-21}{7} = 0$$

$$\Rightarrow \frac{2}{7}x - \frac{6}{7}y + \frac{3}{7}z = \frac{21}{7} = 3.$$

2.Reduece the equation $6x - 2y - 3z + 7 = 0$ of a plane into normal form.

Solution: Equation of the given plane $6x - 2y - 3z + 7 = 0$ and it's normal form

$$\frac{a}{\sqrt{a^2+b^2+c^2}}x + \frac{b}{\sqrt{a^2+b^2+c^2}}y + \frac{c}{\sqrt{a^2+b^2+c^2}}z = \frac{-d}{\sqrt{a^2+b^2+c^2}}$$

$$\sqrt{6^2 + (-2)^2 + (-3)^2} = \sqrt{49} = 7$$

$$\therefore \frac{6x-2y-3z+7}{7} = 0$$

$$\Rightarrow \frac{6}{7}x - \frac{2}{7}y - \frac{3}{7}z = -\frac{7}{7} = -1.$$

3.Find the perpendicular distance from the origin to the plane $3x + 4y - 12z + 65 = 0$.

Solution: The perpendicular distance from the origin to the plane

$ax + by + cz + d = 0$ is $\frac{|d|}{\sqrt{a^2+b^2+c^2}}$

Now the perpendicular distance from the origin to the $3x + 4y - 12z + 65 = 0$ is

$$\frac{|65|}{\sqrt{3^2+4^2+(-12)^2}} = \frac{|65|}{\sqrt{9+16+144}} = \frac{|65|}{\sqrt{169}} = \frac{65}{13} = 5.$$

4.Find the perpendicular distance from the point (-2, 3, 1) to the plane

$2x - 3y - 6z + 5 = 0$.

Solution: The perpendicular distance from the point P (x_1, y_1, z_1) to the plane

$ax + by + cz + d = 0$ is $\frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}$

$\therefore$ The perpendicular distance from the point P $(-2,3,1)$ to the plane

$2x - 3y - 6z + 6 = 0$ is

$$\frac{|2(-2)-3(3)-6(1)+5|}{\sqrt{2^2+(-3)^2+(-6)^2}} = \frac{|-14|}{\sqrt{49}} = \frac{14}{7} = 2$$

5. Find the perpendicular distance from the point P (3, -4, 1) to the plane

$x + y - z + 7 = 0.$

Solution: The perpendicular distance from the point P (x_1, y_1, z_1) to the plane

$$ax + by + cz + d = 0 \text{ is } \frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}$$

∴ The perpendicular distance from the point P(3, -4, 1) to the plane

$x + y - z + 7 = 0$ is

$$\frac{|3-4-1+7|}{\sqrt{1^2+(1)^2+(-1)^2}} = \frac{|5|}{\sqrt{3}} = \frac{5}{\sqrt{3}}$$

6)Find the angle between the planes $2x - y + z - 6 = 0$ and $x + y + 2z + 7 = 0$.

Solution: Angle between the planes

$$a_1x + b_1y + c_1z + d_1 = 0 \text{ and } a_2x + b_2y + c_2z + d_2 = 0 \text{ is}$$

$$\cos\theta = \frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$$

Given that the planes are $2x - y + z - 6 = 0$ and $x + y + 2z + 7 = 0$

$$\text{Now } \cos\theta = \frac{(2)(1)+(-1)(1)+(1)(2)}{\sqrt{4+1+1}\sqrt{1+1+4}} = \frac{3}{\sqrt{6\sqrt{6}}} = \frac{3}{6} = \frac{1}{2} = \cos\frac{\pi}{3}$$

$$\Rightarrow \cos\theta = \cos\frac{\pi}{3} \Rightarrow \theta = \frac{\pi}{3}$$

7)Find the angle between the planes

$$x + 2y + 3z - 5 = 0 \text{ and } 3x + 3y + z + 9 = 0.$$

Solution: Angle between the planes

$$a_1x + b_1y + c_1z + d_1 = 0 \text{ and } a_2x + b_2y + c_2z + d_2 = 0 \text{ is}$$

$$\cos\theta = \frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$$

Given that the planes are $x + 2y + 3z - 5 = 0$ and $3x + 3y + z + 9 = 0$

$$\text{Now } \cos\theta = \frac{(1)(3)+(2)(3)+(3)(1)}{\sqrt{1+4+9}\sqrt{9+9+1}} = \frac{12}{\sqrt{14\sqrt{19}}} = \frac{12}{\sqrt{266}}$$

$$\Rightarrow \cos\theta = \frac{12}{\sqrt{266}} \Rightarrow \theta = \cos^{-1}\frac{12}{\sqrt{266}}.$$

8)Find the equation of a plane through the point (1, 2, -3) and parallel to the plane $2x - 3y + z - 5 = 0$.

Solution: we know the equation of the parallel plane

$$to\ ax + by + cz + d = 0 \text{ is } ax + by + cz + k = 0.$$

The given plane is $2x - 3y + z - 5 = 0$

And it's parallel plane is $2x - 3y + z + k = 0$ where k is a constant.

But it passes through (1, 2, -3)

$$\therefore 2(1) - 3(2) + (-3) + k = 0 \Rightarrow k = 7$$

Equation of parallel plane $2x - 3y + z + 7 = 0.$

9)Find the equation of a plane through the point (4, 0, 1) and parallel to the plane $4x + 3y - 12z + 6 = 0.$

Solution: we know the equation of the parallel plane

$$to\ ax + by + cz + d = 0 \text{ is } ax + by + cz + k = 0.$$

The given plane is $4x + 3y - 12z + 6 = 0$

And it's parallel plane is $4x + 3y - 12z + k = 0$ where k is a constant.

But it passes through (4, 0, 1)

$$\therefore 4(4) + 3(0) - 12(1) + k = 0 \Rightarrow k = -4$$

Equation of parallel plane $4x + 3y - 12z - 4 = 0.$

10)Find the distance between the parallel planes

$$12x - 3y + 4z - 7 = 0 \text{ and } 12x - 3y + 4z + 6 = 0.$$

Solution: The distance between the parallel planes

$$ax + by + cz + d_1 = 0 \ \&\ ax + by + cz + d_2 = 0 \text{ is } \frac{|\, d_1 - d_2 \,|}{\sqrt{a^2 + b^2 + c^2}}$$

Given planes are $12x - 3y + 4z - 7 = 0$ and $12x - 3y + 4z + 6 = 0.$

$$\text{Distance between them} = \frac{|\, 6+7 \,|}{\sqrt{12^2 + 3^2 + 4^2}} = \frac{13}{\sqrt{169}} = \frac{13}{13} = 1$$

11) Find the distance between the parallel planes

$$3x - y + 2z + 4 = 0 \text{ and } 6x - 2y + 4z + 5 = 0.$$

Solution: The distance between the parallel planes

$$ax + by + cz + d_1 = 0 \;\&\; ax + by + cz + d_2 = 0 \text{ is } \frac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}}$$

Given planes are $3x - y + 2z + 4 = 0$ and $6x - 2y + 4z + 5 = 0$.

That is $6x - 2y + 4z + 8 = 0$ and $6x - 2y + 4z + 5 = 0$.

Distance between them $= \frac{|5-4|}{\sqrt{6^2 + (-2)^2 + 4^2}} = \frac{1}{\sqrt{56}} = \frac{1}{2\sqrt{14}}$.

12)Find the equation of a plane passing through the points (2, 2, -1) (3, 4, 2)and (7, 0, 6).

Solution: Let a, b, c are direction ratios of the normal line to the plane.

We know that Equation of a plane passing through P (x_1, y_1, z_1) and whose direction ratios of the normal line are a, b, c is

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0.$$

$\therefore$ Equation passes through the point (2, 2, -1) is

$$a(x - 2) + b(y - 2) + c(z + 1) = 0 \text{ ----------(1)}$$

But it passes through the points (3, 4, 2) and (7, 0, 6)

$$\therefore a(3 - 2) + b(4 - 2) + c(2 + 1) = 0$$

$$a(1) + b(2) + c(3) = 0 \text{ --------------(2)}$$

$$a(7 - 2) + b(0 - 2) + c(6 + 1) = 0$$

$$a(5) + b(-2) + c(7) = 0 \text{ ---------------(3)}$$

To solve the Equations (2) and (3) using the method of cross multiplication

2	3	1	2
-2	7	5	-2

$$\frac{a}{14 - (-6)} = \frac{b}{15 - 7} = \frac{c}{-2 - 10}$$

$$\frac{a}{20} = \frac{b}{8} = \frac{c}{-12} \Rightarrow \frac{a}{5} = \frac{b}{2} = \frac{c}{-3}$$

Put the values in (1)

$$5(x-2)+2(y-2)-3(z+1)=0$$

$$\Rightarrow (5x-10)+(2y-4)-3z-3=0$$

$\Rightarrow 5x+2y-3z-17=0$ is required plane.

13)Find the equation of a plane passing through the points (1, 1, 1) (1, -1, 1)and (-7, -3, -5).

Solution: Let a, b, c are direction ratios of the normal line to the plane.

We know that Equation of a plane passing through P (x_1, y_1, z_1) and whose direction ratios of the normal line are a, b, c is

$$a(x-x_1)+b(y-y_1)+c(z-z_1)=0.$$

∴ Equation passes through the point (1, 1, 1) is

$$a(x-1)+b(y-1)+c(z-1)=0 \text{ ----------(1)}$$

But it passes through the points (1, -1, 1) and (-7, -3, -5)

$$\therefore a(1-1)+b(-1-1)+c(1-1)=0$$

$$a(0)+b(-2)+c(0)=0 \quad \text{--------------(2)}$$

$$a(-7-1)+b(-3-1)+c(-5-1)=0$$

$$a(-8)+b(-4)+c(-6)=0 \text{ ---------------(3)}$$

To solve the Equations (2) and (3) using the method of cross multiplication

-2	0	0	-2
-8	-4	-6	-8

$$\frac{a}{8-(0)}=\frac{b}{0-0}=\frac{c}{0-12}$$

$$\frac{a}{8}=\frac{b}{0}=\frac{c}{-12} \Rightarrow \frac{a}{2}=\frac{b}{0}=\frac{c}{-3}$$

Put the values in (1) $2(x-1)+0(y-1)-3(z-1)=0$

$\Rightarrow 2x-3z+1=0$ is required plane

14)Show that the points (0, 4, 3)(-1, -5, -3)(-2, -2, 1)(1, 1, -1) are coplanar.

Solution: Let a, b, c are direction ratios of the normal line to the plane. First to find the equation of the plane through the points (0, 4, 3) (-1, -5, -3) (-2, -2, 1)

We know that Equation of a plane passing through P (x_1, y_1, z_1) and whose direction ratios of the normal line are a, b, c is

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0.$$

∴ Equation passes through the point (0, 4, 3) is

$$a(x - 0) + b(y - 4) + c(z - 3) = 0 \text{ ----------(1)}$$

But it passes through the points (-1, -5, -3)(-2, -2, 1)

$$\therefore a(-1 - 0) + b(-5 - 4) + c(-3 - 3) = 0$$

$$a(-1) + b(-9) + c(-6) = 0$$

$$a(1) + b(9) + c(6) = 0 \text{ ----------(2)}$$

$$a(-2 - 0) + b(-2 - 4) + c(1 - 3) = 0$$

$$a(-2) + b(-6) + c(-2) = 0$$

$$a(1) + b(3) + c(1) = 0 \text{ ----------(3)}$$

To solve the Equations (2) and (3) using the method of cross multiplication

9	6	1	9
3	1	1	3

$$\frac{a}{9 - 18} = \frac{b}{6 - 1} = \frac{c}{3 - 9}$$

$$\frac{a}{-9} = \frac{b}{5} = \frac{c}{-6} \Rightarrow \frac{a}{9} = \frac{b}{-5} = \frac{c}{6}$$

Put the values in (1)

$$9(x - 0) - 5(y - 4) + 6(z - 3) = 0$$

$$9x - 5y + 6z + 2 = 0 \text{ is the plane.}$$

Now to put the 4^{th} point (1, 1, -1) in $9x - 5y + 6z + 2 = 0$

LHS=$9x - 5y + 6z + 2 = 9(1) - 5(1) + 6(-1) + 2 = 0$= RHS

∴ The given points (0, 4, 3) (-1, -5, -3) (-2, -2, 1) (1, 1, -1) are coplanar.

15)Show that the points (0, -1, -1) (4, 5, 1) (3, 9, 4) (-4, 4, 4) are coplanar.

Solution: Let a, b, c are direction ratios of the normal line to the plane. First to find the equation of the plane through the points (0, -1, -1) (4, 5, 1) (3, 9, 4)

We know that Equation of a plane passing through P (x_1, y_1, z_1) and whose direction ratios of the normal line are a, b, c is

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0.$$

∴ Equation passes through the point (0, 4, 3) is

$$a(x - 0) + b(y + 1) + c(z + 1) = 0 \text{ ----------(1)}$$

But it passes through the points (4, 5, 1) (3, 9, 4)

$$\therefore \; a(4 - 0) + b(5 + 1) + c(1 + 1) = 0$$

$$a(4) + b(6) + c(2) = 0$$

$$a(2) + b(3) + c(1) = 0 \text{ ----------(2)}$$

$$a(3 - 0) + b(9 + 1) + c(4 + 1) = 0$$

$$a(3) + b(10) + c(5) = 0 \text{ ----------(3)}$$

To solve the Equations (2) and (3) using the method of cross multiplication

3	1	2	3
10	5	3	10

$$\frac{a}{15 - 10} = \frac{b}{3 - 10} = \frac{c}{20 - 9}$$

$$\frac{a}{5} = \frac{b}{-7} = \frac{c}{11}$$

Put the values in (1)

$$5(x - 0) - 7(y + 1) + 11(z + 1) = 0$$

$$5x - 7y + 11z + 4 = 0 \text{ is the plane.}$$

Now to put the 4th point (-4, 4, 4) in $5x - 7y + 11z + 4 = 0$

LHS=$5x - 7y + 11z + 4 = 5(-4) - 7(4) + 11(4) + 4 = 0$

∴ The given points (0, -1, -1) (4, 5, 1) (3, 9, 4) (-4, 4, 4) are coplanar.

16) Find the equation of a plane passing through the points

(2, 2, 1) (9, 3, 6) and perpendicular to a plane $2x + 6y + 6z = 9$.

Solution: Let a, b, c are direction ratios of the normal line to the plane.

We know that Equation of a plane passing through P (x_1, y_1, z_1) and whose direction ratios of the normal line are a, b, c is

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0.$$

∴ Equation passes through the point (2, 2, 1) is

$$a(x - 2) + b(y - 2) + c(z - 1) = 0 \text{ ----------(1)}$$

But it passes through the point (9, 3, 6)

$$\therefore \quad a(9 - 2) + b(3 - 2) + c(6 - 1) = 0$$

$$a(7) + b(1) + c(5) = 0 \qquad \text{--------------(2)}$$

Also, it is perpendicular to the plane $2x + 6y + 6z = 9$.

$$\therefore\ a(2) + b(6) + c(6) = 0 \text{ ---------------(3)}$$

To solve the Equations (2) and (3) using the method of cross multiplication

1	5	7	1
6	6	2	6

$$\frac{a}{6 - 30} = \frac{b}{10 - 42} = \frac{c}{42 - 2}$$

$$\frac{a}{-24} = \frac{b}{-32} = \frac{c}{40} \Rightarrow \frac{a}{3} = \frac{b}{4} = \frac{c}{-5}$$

Put the values in (1)

$$3(x - 2) + 4(y - 2) - 5(z - 1) = 0$$

$\Rightarrow 3x + 4y - 5z - 9 = 0$ is required plane

17) Find the equation of a plane passing through the points

(1, -2, 2) (-3, 1, -2) and perpendicular to a plane $x + 2y - 3z = 5$.

Solution: Let a, b, c are direction ratios of the normal line to the plane.

We know that Equation of a plane passing through P (x_1, y_1, z_1) and whose direction ratios of the normal line are a, b, c is

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0.$$

∴ Equation passes through the point (1, -2, 2) is

$$a(x-1) + b(y+2) + c(z-2) = 0 \text{ ----------(1)}$$

But it passes through the point (-3, 1, -2)

$$\therefore \quad a(-3-1) + b(1+2) + c(-2-2) = 0$$

$$a(-4) + b(3) + c(-4) = 0 \qquad \text{--------------(2)}$$

Also, it is perpendicular to the plane $x + 2y - 3z = 5$.

$$\therefore\ a(1) + b(2) + c(-3) = 0 \text{ ---------------(3)}$$

To solve the Equations (2) and (3) using the method of cross multiplication

3	-4	-4	3
2	-3	1	2

$$\frac{a}{-9+8} = \frac{b}{-4-12} = \frac{c}{-8-3}$$

$$\frac{a}{-1} = \frac{b}{-16} = \frac{c}{-11} \quad \Rightarrow \quad \frac{a}{1} = \frac{b}{16} = \frac{c}{11}$$

Put the values in (1)

$$1(x-1) + 16(y+2) + 11(z-2) = 0$$

$$\Rightarrow x + 16y + 11z + 9 = 0 \text{ is required plane}$$

18) Find the equation of a plane passing through the point (4, 4, 0) and perpendicular to the planes $x + 2y + 2z - 5 = 0, 3x + 3y + 2z - 8 = 0$

Solution: Let a, b, c are direction ratios of the normal line to the plane.

We know that Equation of a plane passing through P (x_1, y_1, z_1) and whose direction ratios of the normal line are a, b, c is

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0.$$

∴ Equation passes through the point (4, 4, 0) is

$$a(x-4) + b(y-4) + c(z-0) = 0 \text{ ----------(1)}$$

and it is perpendicular to the plane $x + 2y + 2z - 5 = 0$.

$$a(1) + b(2) + c(2) = 0 \qquad \text{--------------(2)}$$

Also, it is perpendicular to the plane $3x + 3y + 2z - 8 = 0$

$$\therefore\ a(3) + b(3) + c(2) = 0 \text{ ---------------(3)}$$

To solve the Equations (2) and (3) using the method of cross multiplication

2	2	1	2
3	2	3	3

$$\frac{a}{4-6} = \frac{b}{6-2} = \frac{c}{3-6}$$

$$\frac{a}{-2} = \frac{b}{4} = \frac{c}{-3} \Rightarrow \frac{a}{2} = \frac{b}{-4} = \frac{c}{3}$$

Put the values in (1)

$$2(x-4) - 4(y-4) + 3(z-0) = 0$$

$\Rightarrow 2x - 4y + 3z + 8 = 0$ is required plane

19) Find the equation of a plane passing through the point (-1, 3, 2) and perpendicular to the planes $x + 2y + 2z - 5 = 0, 3x + 3y + 2z - 8 = 0$

Solution: Let a, b, c are direction ratios of the normal line to the plane.

We know that Equation of a plane passing through P (x_1, y_1, z_1) and whose direction ratios of the normal line are a, b, c is

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0.$$

$\therefore$ Equation passes through the point (-1, 3, 2) is

$$a(x+1) + b(y-3) + c(z-2) = 0 \text{ ----------(1)}$$

and it is perpendicular to the plane $x + 2y + 2z - 5 = 0$.

$$a(1) + b(2) + c(2) = 0 \quad \text{--------------(2)}$$

Also, it is perpendicular to the plane $3x + 3y + 2z - 8 = 0$

$$\therefore\ a(3) + b(3) + c(2) = 0 \text{ ---------------(3)}$$

To solve the Equations (2) and (3) using the method of cross multiplication

2	2	1	2
3	2	3	3

$$\frac{a}{4-6} = \frac{b}{6-2} = \frac{c}{3-6}$$

$$\frac{a}{-2} = \frac{b}{4} = \frac{c}{-3} \Rightarrow \frac{a}{2} = \frac{b}{-4} = \frac{c}{3}$$

Put the values in (1)

$$2(x+1) - 4(y-3) + 3(z-2) = 0$$

$$\Rightarrow 2x - 4y + 3z + 8 = 0 \quad \text{is required plane.}$$

20)A variable plane meets the coordinate axes at A, B and C and is at a constant distance p from the origin. Show that the locus of the centroid of the triangle ABC is $x^{-2} + y^{-2} + z^{-2} = 9p^{-2}$.

Solution: We know the equation of a plane meets the coordinate axes is the intercepting form and is

$$\frac{x}{\alpha} + \frac{y}{\beta} + \frac{z}{\gamma} = 1$$

whose intercepts are A= $(\alpha, 0, 0)$ B = $(0, \beta, 0)$ and C= $(0, 0, \gamma)$.

Given that p= the perpendicular distance from origin (0, 0, 0) to the plane

$$\frac{x}{\alpha} + \frac{y}{\beta} + \frac{z}{\gamma} = 1$$

The perpendicular distance from the origin to the plane $ax + by + cz + d = 0$ is

$$\frac{|d|}{\sqrt{a^2+b^2+c^2}}$$

$$p = \frac{|-1|}{\sqrt{\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2}}} \Rightarrow p^2 = \frac{1}{\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2}}$$

$$\Rightarrow \frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} = \frac{1}{p^2} \text{ -----------------(1)}$$

To find the locus of centroid of Triangle ABC.

Let P = (x_1, y_1, z_1) is the centroid of Triangle A$(\alpha, 0, 0)$ B$(0, \beta, 0)$ C$(0, 0, \gamma)$

$$= \left(\frac{\alpha+0+0}{3}, \frac{0+\beta+0}{3}, \frac{0+0+\gamma}{3}\right) = \left(\frac{\alpha}{3}, \frac{\beta}{3}, \frac{\gamma}{3}\right)$$

$$\Rightarrow x_1 = \frac{\alpha}{3} \Rightarrow \alpha = 3x_1, \ y_1 = \frac{\beta}{3} \Rightarrow \beta = 3y_1, \text{and } z_1 = \frac{\gamma}{3} \Rightarrow \gamma = 3z_1$$

Put the values in (1)

$$\frac{1}{(3x_1)^2} + \frac{1}{(3y_1)^2} + \frac{1}{(3z_1)^2} = \frac{1}{p^2}$$

$$\frac{1}{9{x_1}^2} + \frac{1}{9{y_1}^2} + \frac{1}{9{z_1}^2} = \frac{1}{p^2}$$

$$\frac{1}{{x_1}^2} + \frac{1}{{y_1}^2} + \frac{1}{{z_1}^2} = \frac{9}{p^2}$$

$${x_1}^{-2} + {y_1}^{-2} + {z_1}^{-2} = 9p^{-2}$$

The locus of the point P = (x_1, y_1, z_1) *is* $x^{-2} + y^{-2} + z^{-2} = 9p^{-2}$.

*21)*A variable plane meets the coordinate axes at A, B and C and is at a constant distance p from the origin. Show that the locus of the centroid of the Tetrahedron OABC is $x^{-2} + y^{-2} + z^{-2} = 16p^{-2}$.

Solution: We know the equation of a plane meets the coordinate axes is the intercepting form and is

$$\frac{x}{\alpha} + \frac{y}{\beta} + \frac{z}{\gamma} = 1$$

whose intercepts are A= (α, 0, 0) B = (0, β, 0) and C= (0, 0, γ).

Given that p = the perpendicular distance from origin (0, 0, 0) to the plane

$$\frac{x}{\alpha} + \frac{y}{\beta} + \frac{z}{\gamma} = 1$$

The perpendicular distance from the origin to the plane $ax + by + cz + d = 0$ *is*

$$\frac{|\,d|}{\sqrt{a^2+b^2+c^2}}$$

$$p = \frac{|-1|}{\sqrt{\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}}} \Rightarrow p^2 = \frac{1}{\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}}$$

$$\Rightarrow \frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} = \frac{1}{p^2} \text{ -----------------(1)}$$

To find the locus of centroid of Tetrahedron OABC.

Let P = (x_1, y_1, z_1) *is the centroid of Tetrahedron*

O (0, 0, 0) A (α, 0, 0) B (0, β, 0) C (0, 0, γ)

$$= \left(\frac{\alpha+0+0+0}{4}, \frac{0+\beta+0+0}{4}, \frac{0+0+\gamma+0}{4}\right) = \left(\frac{\alpha}{4}, \frac{\beta}{4}, \frac{\gamma}{4}\right)$$

$$\Rightarrow x_1 = \frac{\alpha}{4} \Rightarrow \alpha = 4x_1, \quad y_1 = \frac{\beta}{4} \Rightarrow \beta = 4y_1, \textit{ and } z_1 = \frac{\gamma}{4} \Rightarrow \gamma = 4z_1$$

Put the values in (1)

$$\frac{1}{(4x_1)^2} + \frac{1}{(4y_1)^2} + \frac{1}{(4z_1)^2} = \frac{1}{p^2}$$

$$\frac{1}{16{x_1}^2} + \frac{1}{16{y_1}^2} + \frac{1}{16{z_1}^2} = \frac{1}{p^2}$$

$$\frac{1}{{x_1}^2} + \frac{1}{{y_1}^2} + \frac{1}{{z_1}^2} = \frac{16}{p^2}$$

$${x_1}^{-2} + {y_1}^{-2} + {z_1}^{-2} = 16p^{-2}$$

The locus of the point P = (x_1, y_1, z_1) is $x^{-2} + y^{-2} + z^{-2} = 16p^{-2}$.

22)A variable plane meets the coordinate axes at A, B and C and is at a constant distance 3p from the origin. Show that the locus of the centroid of the triangle ABC is $x^{-2} + y^{-2} + z^{-2} = p^{-2}$.

Solution: We know the equation of a plane meets the coordinate axes is the intercepting form and is

$$\frac{x}{\alpha} + \frac{y}{\beta} + \frac{z}{\gamma} = 1$$

whose intercepts are A= $(\alpha, 0, 0)$ B = $(0, \beta, 0)$ and C= $(0, 0, \gamma)$.

Given that 3p = the perpendicular distance from origin (0, 0, 0) to the plane

$$\frac{x}{\alpha} + \frac{y}{\beta} + \frac{z}{\gamma} = 1$$

The perpendicular distance from the origin to the plane $ax + by + cz + d = 0$ is

$$\frac{|d|}{\sqrt{a^2+b^2+c^2}}$$

$$3p = \frac{|-1|}{\sqrt{\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}}} \Rightarrow 9p^2 = \frac{1}{\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}}$$

$$\Rightarrow \frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} = \frac{1}{9p^2} \text{ ------------------(1)}$$

To find the locus of centroid of Triangle ABC.

Let P = (x_1, y_1, z_1) is the centroid of Triangle A$(\alpha, 0, 0)$ B$(0, \beta, 0)$ C$(0, 0, \gamma)$

$$= \left(\frac{\alpha+0+0}{3}, \frac{0+\beta+0}{3}, \frac{0+0+\gamma}{3}\right) = \left(\frac{\alpha}{3}, \frac{\beta}{3}, \frac{\gamma}{3}\right)$$

$$\Rightarrow x_1 = \frac{\alpha}{3} \Rightarrow \alpha = 3x_1,\ \ y_1 = \frac{\beta}{3} \Rightarrow \beta = 3y_1, \text{ and } z_1 = \frac{\gamma}{3} \Rightarrow \gamma = 3z_1$$

Put the values in (1)

$$\frac{1}{(3x_1)^2} + \frac{1}{(3y_1)^2} + \frac{1}{(3z_1)^2} = \frac{1}{9p^2}$$

$$\frac{1}{9{x_1}^2} + \frac{1}{9{y_1}^2} + \frac{1}{9{z_1}^2} = \frac{1}{9p^2}$$

$$\frac{1}{{x_1}^2} + \frac{1}{{y_1}^2} + \frac{1}{{z_1}^2} = \frac{1}{p^2}$$

$$ {x_1}^{-2} + {y_1}^{-2} + {z_1}^{-2} = p^{-2}$$

The locus of the point P = (x_1, y_1, z_1) is $x^{-2} + y^{-2} + z^{-2} = p^{-2}$.

23)A variable plane meets the coordinate axes at A, B and C such that the centroid of the triangle ABC is the point (a, b, c) . Show that the equation of the plane *is* $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 3$

Solution: We know the equation of a plane meets the coordinate axes is the intercepting form and is

$$\frac{x}{\alpha} + \frac{y}{\beta} + \frac{z}{\gamma} = 1 \quad \text{-------------(1)}$$

whose intercepts are A= $(\alpha, 0, 0)$ B = $(0, \beta, 0)$ and C= $(0, 0, \gamma)$.

Given that (a, b, c) is the centroid of Triangle A$(\alpha, 0, 0)$ B$(0, \beta, 0)$ C$(0, 0, \gamma)$

$$= \left(\frac{\alpha+0+0}{3}, \frac{0+\beta+0}{3}, \frac{0+0+\gamma}{3}\right) = \left(\frac{\alpha}{3}, \frac{\beta}{3}, \frac{\gamma}{3}\right)$$

$$\Rightarrow a = \frac{\alpha}{3} \Rightarrow \alpha = 3a, \quad b = \frac{\beta}{3} \Rightarrow \beta = 3b, \text{and } c = \frac{\gamma}{3} \Rightarrow \gamma = 3c$$

Put the values in (1)

$$\frac{x}{3a} + \frac{y}{3b} + \frac{z}{3c} = 1 \quad \Rightarrow \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 3$$

System of Planes

1.Equation of a plane through the intersection of planes

$\pi_1 = a_1x + b_1y + c_1z + d_1 = 0$ and

$\pi_2 = a_2x + b_2y + c_2z + d_2 = 0$ is $\pi_1 + \lambda\, \pi_2 = 0$

i.e. $a_1x + b_1y + c_1z + d_1 + \lambda(a_2x + b_2y + c_2z + d_2) = 0.$

2.Equations of angle bisecting planes between the planes

$\pi_1 = a_1x + b_1y + c_1z + d_1 = 0$ and

$\pi_2 = a_2x + b_2y + c_2z + d_2 = 0$ is

$$\frac{a_1x+b_1y+c_1z+d_1}{\sqrt{a_1^2+b_1^2+c_1^2}} = \pm \frac{a_2x+b_2y+c_2z+d_2}{\sqrt{a_2^2+b_2^2+c_2^2}}$$

Note:1 If the angle between one of the planes and one of the angle bisectors is less than $\frac{\pi}{4}$ then that angle bisector is acute angle bisector and other is obtuse angle bisector.

Note:2 Let $a_1x + b_1y + c_1z + d_1 = 0$ ------(1)

$a_2x + b_2y + c_2z + d_2 = 0$ -------(2) are intersecting planes such that

$d_1d_2 > 0$.

(A) $a_1a_2 + b_1b_2 + c_1c_2 < 0$ then the origin lies in the acute angle between (1) and (2).

(B) $a_1a_2 + b_1b_2 + c_1c_2 > 0$ then the origin lies in the obtuse angle between (1) and (2).

Problems:

1.Find the equation of a plane through the line of intersection of planes $x + y - z - 6 = 0, 2x - 4y + 3z + 5 = 0$ and the point (1, 1, 1).

Solution: We know that Equation of a plane through the intersection of planes

$\pi_1 = 0$ and $\pi_2 = 0$ is $\pi_1 + \lambda\, \pi_2 = 0$ where λ is a constant

$\therefore$ the required plane $x + y - z - 6 + \lambda(2x - 4y + 3z + 5) = 0$ --------(1)

But it passes through the point (1, 1, 1)

$\therefore 1 + 1 - 1 - 6 + \lambda[2(1) - 4(1) + 3(1) + 5] = 0 \Rightarrow -5 + 6\lambda = 0$

$$\therefore \lambda = \frac{5}{6}$$

Put the value in (1) $x + y - z - 6 + \frac{5}{6}(2x - 4y + 3z + 5) = 0$

16x -14y + 9z - 11= 0.

2. Find the equation of a plane through the line of intersection of planes $2x - 7y + 4z - 3 = 0, 3x - 5y + 4z + 11 = 0$ and the point (-2, 1, 3).

Solution: We know that Equation of a plane through the intersection of planes

$$\pi_1 = 0 \text{ and } \pi_2 = 0 \text{ is } \pi_1 + \lambda\,\pi_2 = 0 \text{ where } \lambda \text{ is a constant}$$

∴ the required plane

$$2x - 7y + 4z - 3 + \lambda(3x - 5y + 4z + 11) = 0 \text{ --------(1)}$$

But it passes through the point (-2, 1, 3).

$\therefore 2(-2) - 7(1) + 4(3) - 3 + \lambda[3(-2) - 5(1) + 4(3) + 11] = 0$

$$\therefore \lambda = -2 + \lambda(12) = 0 \Rightarrow \lambda = 1/6$$

Put the value in (1)

$$2x - 7y + 4z - 3 + \frac{1}{6}(3x - 5y + 4z + 11) = 0$$

$$\mathbf{15x - 47y + 28z - 7 = 0}$$

3. Find the equation of a plane through the line of intersection of planes $x + 2y + 3z + 4 = 0, 4x + 3y + 3z + 1 = 0$ and perpendicular to a plane $x + y + z + 9 = 0$,

Solution: We know that Equation of a plane through the intersection of planes

$$\pi_1 = 0 \text{ and } \pi_2 = 0 \text{ is } \pi_1 + \lambda\,\pi_2 = 0 \text{ where } \lambda \text{ is a constant}$$

∴ the required plane

$$\mathbf{x + 2y + 3z + 4} + \lambda(\mathbf{4x + 3y + 3z + 1}) = 0 \text{ -------(1)}$$

$$(1+4\lambda)x + (2+3\lambda)y + (3 + 3\lambda)z + (4 + \lambda) = 0$$

But it is perpendicular to $x + y + z + 9 = 0$

$$\therefore a_1a_2 + b_1b_2 + c_1c_2 = 0$$

$(1+4\lambda).1 + (2+3\lambda).1 + (3 + 3\lambda).1 = 0 \Rightarrow 6 + 10\lambda = 0 \Rightarrow \lambda = -3/5$

Put the value in (1)

$$x + 2y + 3z + 4 - \frac{3}{5}(4x + 3y + 3z + 1) = 0$$

$$\mathbf{-7x + y + 6z + 17 = 0}$$

4.Find the bisecting planes of acute angle between the planes

$3x - y + 2z + 3 = 0, \; 3x - 2y + \; 6z + 8 = 0$ and distinguish them.

Solution: We know that equations of angle bisecting planes between the planes

$\pi_1 = a_1x + b_1y + \; c_1$z $+d_1$= 0 and

$\pi_2 = a_2x + b_2y + \; c_2$z $+d_2$= 0 is

$$\frac{a_1x+b_1y+c_1z+d_1}{\sqrt{a_1{}^2+b_1{}^2+c_1{}^2}} = \pm\frac{a_2x+b_2y+c_2z+d_2}{\sqrt{a_2{}^2+b_2{}^2+c_2{}^2}}$$

Given planes are $3x - y + 2z + 3 = 0$ -------(1)

$3x - 2y + \; 6z + 8 = 0$ --------(2)

$$\frac{3x-y+2z+3}{\sqrt{4+1+4}} = \pm\frac{3x-2y+6z+8}{\sqrt{9+4+36}}$$

$$\Rightarrow \frac{3x-y+2z+3}{3} = \pm\frac{3x-2y+6z+8}{7}$$

$\Rightarrow 7(3x - y + 2z + 3) = \pm 3(3x - 2y + \; 6z + 8)$

$\Rightarrow (21x - 7y + 14z + 21) = \pm(9x - 6y + \; 18z + 24)$

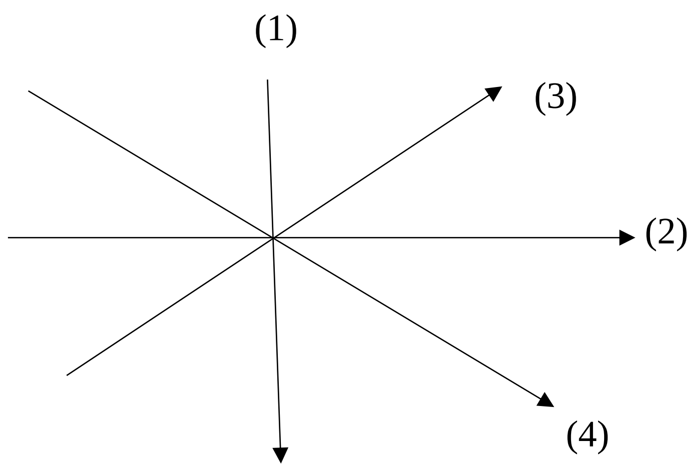
$\Rightarrow \quad (21x - 7y + 14z + 21) = +(9x - 6y + \; 18z + 24)$

Or $(21x - 7y + 14z + 21) = -(9x - 6y + \; 18z + 24)$

$\Rightarrow \; 30x - 13y + 32z + 45 = 0 \; or \; 12x - y - 4z - 3 = 0$

$12x - y - 4z - 3 = 0$ ---------(3)

$30x - 13y + 32z + 45 = 0$ ------(4)



To find the angle between the planes (1) and (3)

$$3x - y + 2z + 3 = 0 \quad \text{-------(1)}$$

$$12x - y - 4z - 3 = 0 \quad \text{---------(3)}$$

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1{}^2 + b_1{}^2 + c_1{}^2}\sqrt{a_2{}^2 + b_2{}^2 + c_2{}^2}}$$

$$\cos\theta = \frac{3(12) + (-1)(-1) + (2)(-4)}{\sqrt{9+1+4}\sqrt{144+1+16}}$$

$$= \frac{19}{\sqrt{161}\sqrt{14}} = \frac{19}{\sqrt{2254}} = \frac{19}{47.47} > \frac{1}{\sqrt{2}} = \cos 45$$

$$\theta > \pi/4$$

$\therefore$ the plane (3) bisects the plane (1) with obtuse angle and hence the plane (4) bisects the plane (2) with acute angle i.e. $30x - 13y + 32z + 45 = 0$

Given planes $3x - y + 2z + 3 = 0$ -------(1)

$$3x - 2y + 6z + 8 = 0 \text{ --------(2)}$$

As $d_1d_2 = 3 \times 8 = 24 > 0$

And $a_1a_2 + b_1b_2 + c_1c_2 = (3)(3) + (-1)(-1) + (2)(6) = 22 > 0$ then the origin lies in the obtuse angle between (1) and (2).

5.Find the bisecting planes of acute angle between the planes $3x - 2y + 6z + 2 = 0, -2x + y - 2z - 2 = 0$ and distinguish them.

Solution: We know that equations of angle bisecting planes between the planes

$$\pi_1 = a_1x + b_1y + c_1z + d_1 = 0 \text{ and}$$

$$\pi_2 = a_2x + b_2y + c_2z + d_2 = 0 \text{ is}$$

$$\frac{a_1x+b_1y+c_1z+d_1}{\sqrt{a_1{}^2+b_1{}^2+c_1{}^2}} = \pm\frac{a_2x+b_2y+c_2z+d_2}{\sqrt{a_2{}^2+b_2{}^2+c_2{}^2}}$$

Given planes are $3x - 2y + 6z + 2 = 0$ -------(1)

$$2x - y + 2z + 2 = 0 \text{ --------(2)}$$

$$\frac{2x-y+2z+2}{\sqrt{4+1+4}} = \pm \frac{3x-2y+6z+2}{\sqrt{9+4+36}}$$

$$\Rightarrow \frac{2x-y+2z+2}{3} = \pm \frac{3x-2y+6z+2}{7}$$

$$\Rightarrow 7(2x-y+2z+2) = \pm 3(3x-2y+6z+2)$$

$$\Rightarrow (14x-7y+14z+14) = \pm(9x-6y+18z+6)$$

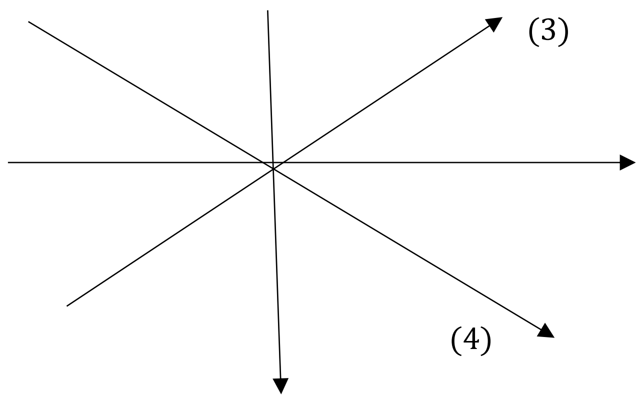
$$\Rightarrow \quad (14x-7y+14z+14) = +(9x-6y+18z+6)$$

$$\text{Or } (14x-7y+14z+14) = -(9x-6y+18z+6)$$

$$\Rightarrow 5x-y-4z+8=0 \text{ or } 23x-13y+32z+20=0$$

$$5x-y-4z+8=0 \quad \text{---------}(3)$$

$$23x-13y+32z+20=0 \quad \text{------}(4)$$



To find the angle between the planes (1) and (3)

$$3x-2y+6z+2=0 \quad \text{-------}(1)$$

$$5x-y-4z+8=0 \quad \text{---------}(3)$$

$$\cos\theta = \frac{|a_1a_2+b_1b_2+c_1c_2|}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$$

$$\cos\theta = \frac{|3(5)+(-2)(-1)+(6)(-4)|}{\sqrt{9+4+36}\sqrt{25+1+16}}$$

$$\text{Cos }\theta = \frac{7}{\sqrt{49}\sqrt{42}} = \frac{1}{\sqrt{42}}$$

Now $\quad tan^2\theta = sec^2\theta - 1 = 42 - 1 = 41$

$$\text{Tan}\,\theta = \sqrt{41} > 1 = \tan \pi/4 \qquad \theta > \pi/4$$

$\therefore$ the plane (3) bisects the plane (1) with obtuse angle and hence the plane (4) bisects the plane (2) with acute angle i.e., $23x - 13y + 32z + 20 = 0$

Given planes $3x - 2y + 6z + 2 = 0$ -------(1)

$2x - y + 2z + 2 = 0$ --------(2)

As $d_1 d_2 = 2 \times 2 = 4 > 0$

And $a_1 a_2 + b_1 b_2 + c_1 c_2 = (3)(2) + (-2)(-1) + (6)(2) = 20 > 0$

then the Origin lies in the obtuse angle plane i.e., $5x - y - 4z + 8 = 0$ between (1) and (2).

6.Find the bisecting planes of acute angle between the planes

$3x - 6y + 2z + 5 = 0, 4x - 12y + 3z - 3 = 0$ and distinguish them.

Solution: We know that equations of angle bisecting planes between the planes

$\pi_1 = a_1 x + b_1 y + c_1 z + d_1 = 0$ and

$\pi_2 = a_2 x + b_2 y + c_2 z + d_2 = 0$ is

$$\frac{a_1 x + b_1 y + c_1 z + d_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = \pm \frac{a_2 x + b_2 y + c_2 z + d_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

Given planes are $3x - 6y + 2z + 5 = 0$ -------(1)

$4x - 12y + 3z - 3 = 0$ --------(2)

$$\frac{3x - 6y + 2z + 5}{\sqrt{9 + 36 + 4}} = \pm \frac{4x - 12y + 3z - 3}{\sqrt{16 + 144 + 9}}$$

$$\Rightarrow \frac{3x - 6y + 2z + 5}{7} = \pm \frac{4x - 12y + 3z - 3}{13}$$

$$\Rightarrow 13(3x - 6y + 2z + 5) = \pm 7(4x - 12y + 3z - 3)$$

$$\Rightarrow (39x - 78y + 26z + 65) = \pm(28x - 84y + 21z - 21)$$

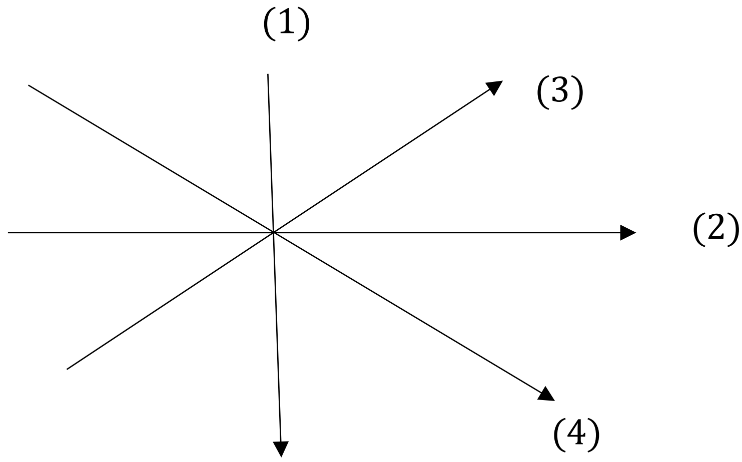
$$\Rightarrow (39x - 78y + 26z + 65) = +(28x - 84y + 21z - 21)$$

$$\text{or } (39x - 78y + 26z + 65) = -(28x - 84y + 21z - 21)$$

$$\Rightarrow 11x + 6y + 5z + 86 = 0 \text{ or } 67x - 162y + 47z + 44 = 0$$

$11x + 6y + 5z + 86 = 0$ ---------(3)

$$67x - 162y + 47z + 44 = 0 \text{ ------(4)}$$



To find the angle between the planes (1) and (3)

$$3x - 6y + 2z + 5 = 0 \text{ -------(1)}$$

$$11x + 6y + 5z + 86 = 0 \text{ ---------(3)}$$

$$\cos\theta = \frac{|a_1a_2 + b_1b_2 + c_1c_2|}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

$$\cos\theta = \frac{|3(11) + (-6)(6) + (2)(5)|}{\sqrt{9 + 36 + 4}\sqrt{121 + 36 + 25}}$$

$$\text{Cos}\,\theta = \frac{7}{\sqrt{49}\sqrt{182}} = \frac{1}{\sqrt{182}}$$

Now $tan^2\theta = sec^2\theta - 1 = 182 - 1 = 181$

$$\text{Tan}\,\theta = \sqrt{181} > 1 = tan\,\pi/4 \qquad \theta > \pi/4$$

∴ the plane (3) bisects the plane (1) with obtuse angle and hence the plane (4) bisects the plane (2) with acute angle i.e., $67x - 162y + 47z + 44 = 0$

Given planes are $3x - 6y + 2z + 5 = 0$ -------(1)

$-4x + 12y - 3z + 3 = 0$ --------(2)

As $d_1d_2 = 5 \times 3 = 15 > 0$

And $a_1a_2 + b_1b_2 + c_1c_2 = (3)(-4) + (-6)(12) + (2)(-3) = -90 < 0$

then the Origin lies in the acute angle plane i.e., between (1) and (2)

$$67x - 162y + 47z + 44 = 0.$$

7.Find the bisecting planes of acute angle between the planes

$-x + 2y - 2z + 19 = 0,\ 4x - 3y + 12z + 3 = 0$ and distinguish them.

Solution: We know that equations of angle bisecting planes between the planes

$\pi_1 = a_1x + b_1y + c_1z + d_1 = 0$ and

$\pi_2 = a_2x + b_2y + c_2z + d_2 = 0$ is

$$\frac{a_1x+b_1y+c_1z+d_1}{\sqrt{a_1^2+b_1^2+c_1^2}} = \pm \frac{a_2x+b_2y+c_2z+d_2}{\sqrt{a_2^2+b_2^2+c_2^2}}$$

Given planes are $-x + 2y - 2z + 19 = 0$ -------(1)

$4x - 3y + 12z + 3 = 0$ --------(2)

$$\frac{-x+2y-2z+19}{\sqrt{1+4+4}} = \pm \frac{4x-3y+12z+3}{\sqrt{16+9+144}}$$

$$\Rightarrow \frac{-x+2y-2z+19}{3} = \pm \frac{4x-3y+12z+3}{13}$$

$$\Rightarrow 13(-x + 2y - 2z + 19) = \pm 3(4x - 3y + 12z + 3)$$

$$\Rightarrow (-13x + 26y - 26z + 247) = \pm(12x - 9y + 36z + 9)$$

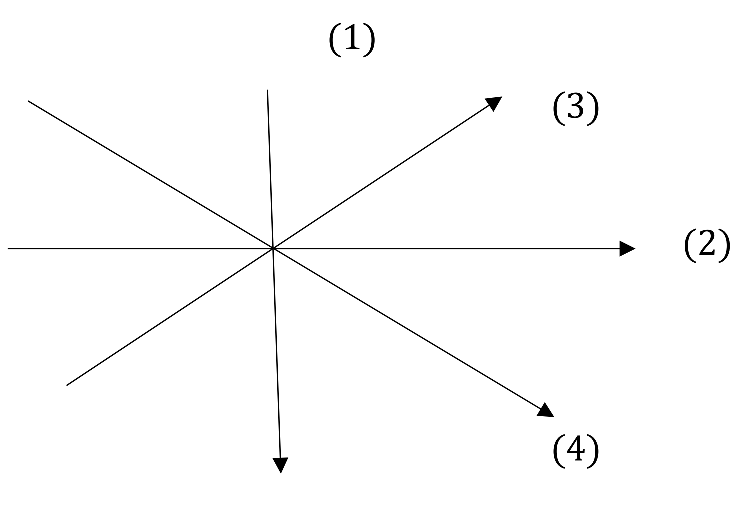
$$\Rightarrow \quad (-13x + 26y - 26z + 247) = +(12x - 9y + 36z + 9)$$

or $(-13x + 26y - 26z + 247) = -(12x - 9y + 36z + 9)$

$$\Rightarrow -25x + 35y - 62z + 238 = 0 \ or -x - 17y + 10z + 256 = 0$$

$-x - 17y + 10z + 256 = 0$ ---------(3)

$-25x + 35y - 62z + 238 = 0$ ------(4)



To find the angle between the planes (1) and (3)

$$-x + 2y - 2z + 19 = 0 \ = 0 \text{ -------}(1)$$

$$-x - 17y + 10z + 256 = 0 \text{ ---------}(3)$$

$$\cos\theta = \frac{|a_1a_2+b_1b_2+c_1c_2\,|}{\sqrt{a_1{}^2+b_1{}^2+c_1{}^2}\,\sqrt{a_2{}^2+b_2{}^2+c_2{}^2}}$$

$$\cos\theta = \frac{|(-1)(-1)+(2)(-17)+(-2)(10)|}{\sqrt{1+4+4}\,\sqrt{1+289+100}}$$

$$\text{Cos}\,\theta = \frac{|-53|}{3\sqrt{390}} = \frac{53}{3\sqrt{390}}$$

Now $tan^2\theta = sec^2\theta - 1 = \frac{2809}{3510} - 1 = \frac{701}{3510}$

$$\text{Tan}\,\theta = \sqrt{\frac{701}{3510}} < 1 = tan\,\pi/4 \qquad \theta < \pi/4$$

∴ the plane (3) bisects the plane (1) with acute angle and hence the plane (4) bisects the plane (2) with obtuse angle i.e., 0

Given planes

$$-x + 2y - 2z + 19 = 0 \text{ -------}(1)$$

$$4x - 3y + 12z + 3 = 0 \text{ ---------}(2)$$

As $d_1d_2 = 19 \times 3 = 57 > 0$

And $a_1a_2 + b_1b_2 + c_1c_2 = (-1)(4) + (2)(-3) + (-2)(12) = -34 < 0$

then the Origin lies in the acute angle plane i.e., between (1) and (2).

$$-x - 17y + 10z + 256 = 0$$

Pair of Planes

1.The joint equation of two planes is a pair of planes.

2.The general equation of pair of planes passes through origin is

$$ax^2 + by^2 + cz^2 + 2hxy + 2fyz + 2gzx = 0$$

3. The general equation of pair of planes does not passes through origin is

$$ax^2 + by^2 + cz^2 + 2hxy + 2fyz + 2gzx + 2ux + 2vy + 2wz + d = 0$$

4.Conditions for an equation $ax^2 + by^2 + cz^2 + 2hxy + 2fyz + 2gzx = 0$

represents pair of planes are

(i)$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = 0.$

(ii)$f^2 \geq bc,\ g^2 \geq ac, h^2 \geq ab.$

5.Angle between the pair of planes

$ax^2 + by^2 + cz^2 + 2hxy + 2fyz + 2gzx = 0$ is

$$\text{Cos}\ \theta = \frac{|a+b+c|}{\sqrt{(a+b+c)^2+4(f^2+g^2+h^2-ab-bc-ca)}}$$

(i)Condition for a pair of plane represents perpendicular lines $a + b + c = 0$

(ii)Condition for a pair of plane represents parallel lines

$$f^2 = bc,\ g^2 = ac, h^2 = ab.$$

Problems:

1.Show that the equation $2x^2 - 6y^2 - 12z^2 + xy + 18yz + 2zx = 0$ represents a pair of planes, and find the angle between them.

Solution: Compare the equation $2x^2 - 6y^2 - 12z^2 + xy + 18yz + 2zx = 0$

With the equation $ax^2 + by^2 + cz^2 + 2hxy + 2fyz + 2gzx = 0$

a = 2 b = -6 c = -12

2h = 1 ⇒ h=1/2 2f =18 ⇒f = 9 2g = 2 ⇒ g =1

LHS= $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$

= 144 +(1)(9)(1)-2(81)+6(1)+12(1/4)

=144 + 9 - 162 + 6 + 3 = 0 RHS

The equation represents pair of planes

Angle between them

$$\text{Cos}\,\theta = \frac{|\,a+b+c|}{\sqrt{(a+b+c)^2+4(f^2+g^2+h^2-ab-bc-ca)}}$$

$$= \frac{|-16|}{\sqrt{(-16)^2+4[81+1+\frac{1}{4}-(-12)-(72)-(-24)]}}$$

$$= \frac{16}{\sqrt{256+4[81+1+\frac{1}{4}+12-72+24]}}$$

$$= \frac{16}{\sqrt{256+4[46+\frac{1}{4}]}} = \frac{16}{\sqrt{441}} = \frac{16}{21} \quad \Rightarrow \theta = \cos^{-1}\frac{16}{21}$$

2.Show that the equation $2x^2 - 2y^2 + 4z^2 + 3xy + 2yz + 6zx = 0$ represents a pair of planes, and find the angle between them.

Solution: Compare the equation $2x^2 - 2y^2 + 4z^2 + 3xy + 2yz + 6zx = 0$

With the equation $ax^2 + by^2 + cz^2 + 2hxy + 2fyz + 2gzx = 0$

a = 2 b = -2 c = 4

2h = 3 ⇒ h=3/2 2f =2 ⇒ f = 1 2g = 6 ⇒ g =3

LHS= $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$

= -16 +(3)(3)(1)-2(1)+2(9)-4(9/4)

=-16 + 9 - 2 + 18 - 9 = 0 RHS

The equation represents pair of planes

Angle between them

$$\text{Cos}\,\theta = \frac{|\,a+b+c|}{\sqrt{(a+b+c)^2+4(f^2+g^2+h^2-ab-bc-ca)}}$$

$$= \frac{|4|}{\sqrt{(4)^2+4[1+9+\frac{9}{4}-(-4)-(-8)-(8)]}}$$

$$= \frac{4}{\sqrt{16+4[10+\frac{9}{4}+4]}}$$

$$= \frac{4}{\sqrt{16+56+9]}} = \frac{4}{\sqrt{81}} = \frac{4}{9} \quad \Rightarrow \ \theta = \cos^{-1}\frac{4}{9}$$

3. Show that the equation $6x^2 + 4y^2 - 10z^2 - 11xy + 3yz + 4zx = 0$ represents a pair of planes, and find the angle between them.

Solution: Compare the equation

$$6x^2 + 4y^2 - 10z^2 - 11xy + 3yz + 4zx = 0$$

With the equation $ax^2 + by^2 + cz^2 + 2hxy + 2fyz + 2gzx = 0$

a = 6 b = 4 c = 10

2h = -11 ⇒ h= -11/2, 2f =3 ⇒ f = 3/2 2g = 4 ⇒ g =2

LHS= $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$

= - 240+(3)(2)(-11/2) - 6(9/4)-4(4) +10(121/4)

= - 240 – 33 -27/2 -16 +605/2

= - 289 + $\frac{-27+605}{2}$ = -289 + $\frac{578}{2}$ $= -289 + 289 = 0$ =RHS

The equation represents pair of planes

Angle between them

$$\text{Cos}\ \theta = \frac{|a+b+c|}{\sqrt{(a+b+c)^2+4(f^2+g^2+h^2-ab-bc-ca)}}$$

$$= \frac{|0|}{\sqrt{denominator}} = 0 = \cos\frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{2}$$

4. Show that

$$x^2 + 4y^2 + 9z^2 + 4xy - 12yz - 6zx + 5x + 10y - 15z + 6 = 0$$

represents a pair of parallel planes and find the distance between them.

Solution: Given that

$$x^2 + 4y^2 + 9z^2 + 4xy - 12yz - 6zx + 5x + 10y - 15z + 6 = 0$$

Let a = 1, b=4, c = 9,

2h =4 ⇒ h = 2, 2f = -12 ⇒ f =-6, 2g = -6 ⇒ g = -3

To verify the conditions $f^2 = bc,\ g^2 = ac, h^2 = ab.$

As $f^2 = (-6)^2 = 36 = 4(9) = bc$.

$g^2 = (-3)^2 = 9 = 1(9) = ac$

$h^2 = (2)^2 = 4 = 1(4) = ab$

$\therefore$ The equation represents a pair of parallel planes.

Consider: $x^2 + 4y^2 + 9z^2 + 4xy - 12yz - 6zx$

$= x^2 + (2y)^2 + (3z)^2 + 2x(2y) - 2(2y)(3z) - 2(3z)x$

$= (x + 2y - 3z)^2$

Hence

$$x^2 + 4y^2 + 9z^2 + 4xy - 12yz - 6zx + 5x + 10y - 15z + 6$$

$$\cong (x + 2y - 3z + l)(x + 2y - 3z + k)$$

Comparing the coefficients of x and constant term

$l + k = 5$ -------(1) and $lk = 6$ ------(2)

We know that $(l - k)^2 = (l + k)^2 - 4lk$

$(l - k)^2 = (5)^2 - 4(6) = 25 - 24 = 1$

$\Rightarrow l - k = 1$ ---------(3)

To solve (1)&(3) $l = 3, k = 2$

Equations of parallel planes $(x + 2y - 3z + 3) = 0$ $(x + 2y - 3z + 2) = 0$

The distance between the parallel planes $= \frac{|\, d_1 - d_2 \,|}{\sqrt{a^2 + b^2 + c^2}}$

$$= \frac{|\, 2 - 3 \,|}{\sqrt{1^2 + 2^2 + (-3)^2}} = \frac{1}{\sqrt{14}}$$

5. Show that

$$x^2 + 4y^2 + 4z^2 + 4xy + 8yz + 4zx - 9x - 18y - 18z + 18 = 0$$

represents a pair of parallel planes and find the distance between them.

Solution: Given that

$$x^2 + 4y^2 + 4z^2 + 4xy + 8yz + 4zx - 9x - 18y - 18z + 18 = 0$$

Let a = 1, b=4, c = 4,

2h =4 ⇒ h = 2, 2f = 8 ⇒ f = 4, 2g = 4 ⇒ g = 2

To verify the conditions $f^2 = bc,\ g^2 = ac, h^2 = ab$.

As $f^2 = (4)^2 = 16 = 4(4) = bc$.

$g^2 = (2)^2 = 4 = 1(4) = ac$

$h^2 = (2)^2 = 4 = 1(4) = ab$

$\therefore$ The equation represents a pair of parallel planes.

Consider: $x^2 + 4y^2 + 4z^2 + 4xy + 8yz + 4zx$

$$= x^2 + (2y)^2 + (2z)^2 + 2x(2y) + 2(2y)(2z) + 2(z)x$$

$$= (x + 2y + 2z)^2$$

Hence $x^2 + 4y^2 + 4z^2 + 4xy + 8yz + 4zx - 9x - 18y - 18z + 18$

$$\cong (x + 2y + 2z + l)(x + 2y + 2z + k)$$

Comparing the coefficients of x and constant term

$l + k = -9$ -------(1) and $lk = 18$ ------(2)

We know that $(l - k)^2 = (l + k)^2 - 4lk$

$$(l - k)^2 = (-9)^2 - 4(18) = 81 - 72 = 9$$

$\Rightarrow l - k = 3$ ---------(3)

To solve (1)&(3) $l = -3, k = -6$

Equations of parallel planes $(x + 2y + 2z - 3) = 0\ (x + 2y + z - 6) = 0$

The distance between the parallel planes $= \frac{|\ d_1 - d_2\ |}{\sqrt{a^2 + b^2 + c^2}}$

$$= \frac{|-3+6|}{\sqrt{1^2 + 2^2 + 2^2}} = \frac{3}{\sqrt{9}} = 1$$

All the best -BSR

2. Right Line

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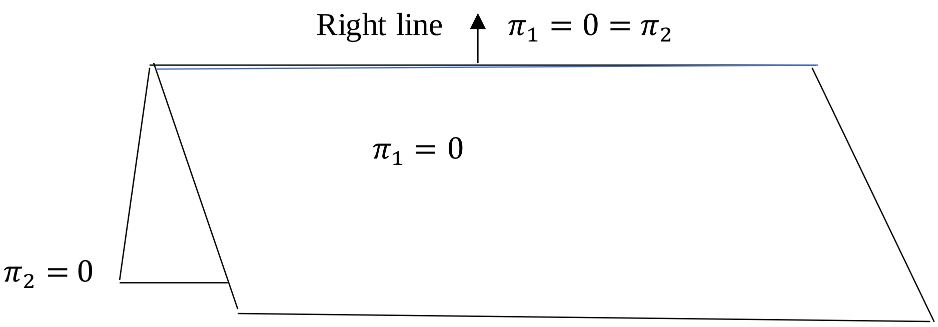
Definition:

The right line is a line made by the intersection of two planes and is also called straight line or simply line.

1. If $\pi_1 = a_1x + b_1y + c_1z + d_1 = 0$ and $\pi_2 = a_2x + b_2y + c_2z + d_2 = 0$ are two planes then the equation of the right line is $\pi_1 = 0 = \pi_2$ i.e., $a_1x + b_1y + c_1z + d_1 = 0 = a_2x + b_2y + c_2z + d_2$. It is called non- symmetric form of a line.



2. Equation of line through the point A (x_1, y_1, z_1) whose direction ratios are l, m, n is

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$$

And is also called symmetric form of a right line.

3.In the symmetric form of a line let

$$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n} = r$$

$$x - x_1 = lr, \quad y - y_1 = mr, \quad z - z_1 = nr$$

$$x = lr + x_1, \quad y = mr + y_1, \quad z = nr + z_1$$

Is a general point on the line and is also a point P when it is of distance r from the point A = (x_1, y_1, z_1).

$$P = (lr + x_1, mr + y_1, nr + z_1)$$

4.Equation of line passing through the points A = (x_1, y_1, z_1) B = (x_2, y_2, z_2) is

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

Problems:

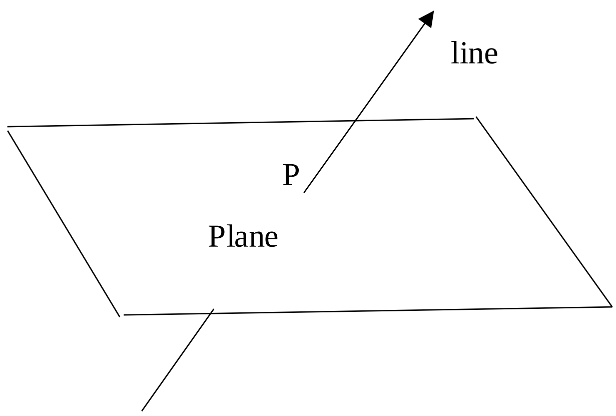
1. Find the point of intersection of line $\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{12}$ and the plane $x - 2y + z = 20$.

Solution: Let $\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{12} = r$

$$\Rightarrow x - 2 = 3r, y + 1 = 4r, z - 2 = 12r$$

$$\Rightarrow x = 3r + 2, y = 4r - 1, z = 12r + 2.$$

Is a general point on the line. But it intersects the plane $x - 2y + z = 20$



∴ Point P = $(3r + 2, 4r - 1, 12r + 2)$ lies on $x - 2y + z = 20$

$$(3r + 2) - 2(4r - 1) + (12r + 2) = 20$$

⇒ 7r + 6 = 20 ⇒ 7r = 14 ⇒ r = 2.

The point of intersection P =$(3(2) + 2, 4(2) - 1, 12(2) + 2)$ = (8, 7, 26)

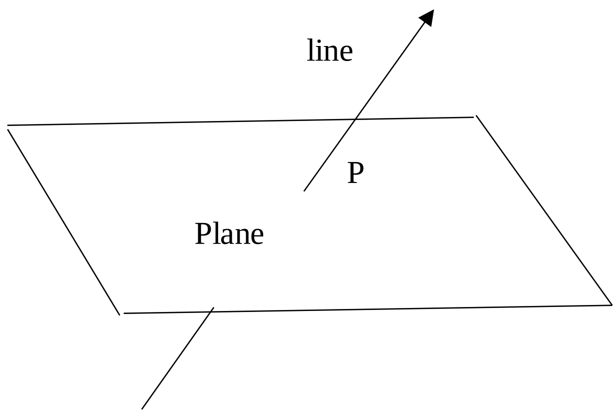
2. Find the point of intersection of line $\frac{x+1}{1} = \frac{y+3}{3} = \frac{z+2}{-2}$ and the plan $3x + 4y + 5z = 5$.

Solution: Let $\frac{x+1}{1}=\frac{y+3}{3}=\frac{z+2}{-2}=r$

$$\Rightarrow x+1=r, y+3=3r, z+2=-2r$$

$$\Rightarrow x=r-1, y=3r-3, z=-2r-2.$$

Is a general point on the line. But it intersects the plane $3x+4y+5z=5$



$\therefore$ Point P = $(r-1, 3r-3, -2r-2)$ lies on $3x+4y+5z=5$

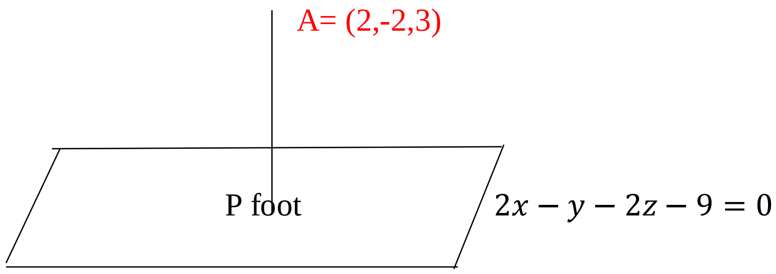
$$3(r-1)+4(3r-3)+5(-2r-2)=5$$

$5r-25=5 \Rightarrow 5r = 30 \Rightarrow$ $r = 6$

The point of intersection P = $(6-1, 3(6)-3, -2(6)-2)$ = $(5, 15, -14)$

3.Find the foot of the perpendicular from (2,-2,3) to the plane $2x-y-2z-9=0$

Solution:



The Drs of the perpendicular line = Drs of the plane $2x-y-2z-9=0$

= 2, -1, -2

$\therefore$Equation of the line through the point A= (2,-2,3) with Drs 2,-1,-2 is

$$\frac{x-2}{2} = \frac{y+2}{-1} = \frac{z-3}{-2} = r\ (say)$$

$\Rightarrow x = 2r+2, y = -r-2, z = -2r+3$

$$Let\ P = (\ 2r+2, -r-2, -2r+3)$$

But the line intersects the plane $2x - y - 2z - 9 = 0$

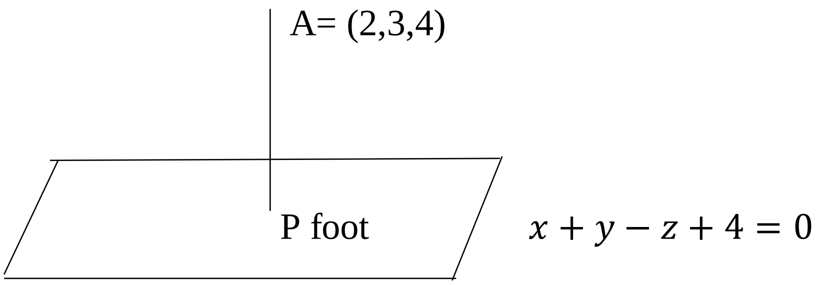
$\therefore 2(2r+2) - (-r-2) - 2(-2r+3) - 9 = 0$

$$9r - 9 = 0 \Rightarrow r = 1$$

Foot of the perpendicular P =$(\ 2+2, -1-2, -2+3) = (4, -3, 1)$

4.Find the foot of the perpendicular from (2,3,4) to the plane $x + y - z + 4 = 0$

Solution:



The Drs of the perpendicular line = Drs of the plane $x + y - z + 4 = 0$

= 1, 1, -1

∴Equation of the line through the point A= (2, 3, 4) with Drs 1,1,-1 is

$$\frac{x-2}{1} = \frac{y-3}{1} = \frac{z-4}{-1} = r\ (say)$$

$$\Rightarrow x = r+2, y = r+3, z = -r+4$$

$$Let\ P = (\ r+2, r+3, -r+4)$$

But the line intersects the plane $x + y - z + 4 = 0$

$$\therefore r + 2 + r + 3 + r - 4 + 4 = 0$$

$$3r + 5 = 0 \Rightarrow r = -5/3$$

Foot of the perpendicular

P =$(\ r+2, r+3, -r+4) = (-5/3+2, -5/3+3, 5/3+4)$

= (1/6, 4/3,17/3)

5.Find the value of k if the lines

$\frac{x-1}{-3} = \frac{y-2}{2k} = \frac{z-3}{2}$, $\frac{x-1}{3k} = \frac{y-5}{1} = \frac{z-6}{-5}$ *are perpendicular.*

Solution: Condition for the two lines are perpendicular

$$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$$

Now $(l_1, m_1, n_1) = (-3, 2k, 2)$ and $(l_2, m_2, n_2) = (3k, 1, -5)$

$\therefore$ (-3) (3k) + (2k) (1) + (2) (-5) = 0 $\Rightarrow$ -9k +2k-10 = 0 $\Rightarrow$ $k = \frac{-10}{7}$.

6.Find the condition that the lines $x = az + b, y = cz + d$ *and* $x = a_1 y + b_1, y = c_1 z + d_1$ are perpendicular.

Solution: Given that $x = az + b, y = cz + d$

$$\Rightarrow x - b = az , y - d = cz$$

$$\Rightarrow \frac{x-b}{a} = z , \frac{y-d}{c} = z$$

$$\Rightarrow \frac{x-b}{a} = \frac{y-d}{c} = \frac{z}{1}$$

Similarly, the second line $x = a_1 y + b_1, y = c_1 z + d_1$

Can be written as $\frac{x-b_1}{a_1} = \frac{y-d_1}{c_1} = \frac{z}{1}$

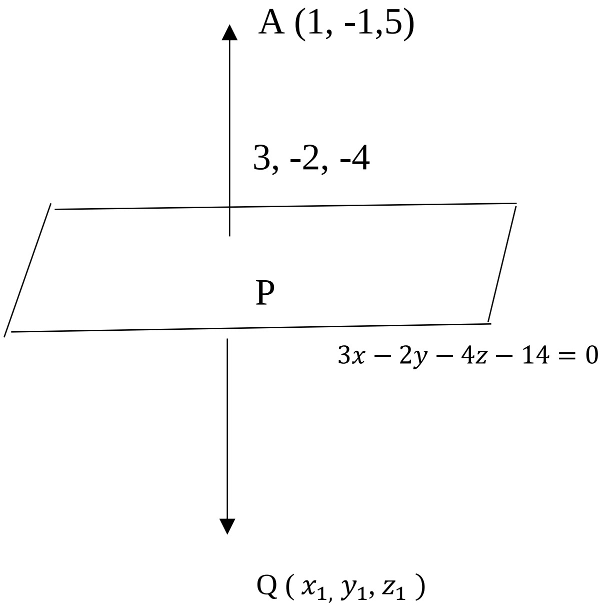
But the lines are perpendicular

$$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$$

$$a a_1 + c c_1 + 1 = 0.$$

7.Find the image of the point (1, -1, 5) in the plane $3x - 2y - 4z - 14 = 0$.

Solution



Let A = (1, -1, 5) and Q = (x_1, y_1, z_1) is the image of A

Direction ratios of AQ = Direction rations of the plane $3x - 2y - 4z - 14 = 0$

= 3, -2, -4

Equation of the line AQ through A= (1, -1, 5) with d.rs 3, -2, -4

$$\frac{x-1}{3} = \frac{y+1}{-2} = \frac{z-5}{-4} = r \ say$$

$$\Rightarrow x = 3r + 1, y = -2r - 1, z = -4r + 5.$$

Let P = $(3r + 1, -2r - 1, -4r + 5)$

But the line meets the plane $3x - 2y - 4z - 14 = 0$

$$\therefore 3(3r + 1) - 2(-2r - 1) - 4(-4r + 5) - 14 = 0$$

$$29r - 29 = 0 \Rightarrow r = 1$$

$\therefore$ P =$(3(1) + 1, -2(1) - 1, -4(1) + 5)$ = (4, -3, 1)

Clearly P = The mid-point of A = (1, -1, 5) and Q = (x_1, y_1, z_1)

$$(4, -3, 1) = \left(\frac{x_1+1}{2}, \frac{y_1+(-1)}{2}, \frac{z_1+5}{2} \right)$$

$$\Rightarrow \frac{x_1+1}{2} = 4 \Rightarrow x_1 = 8 - 1 = 7$$

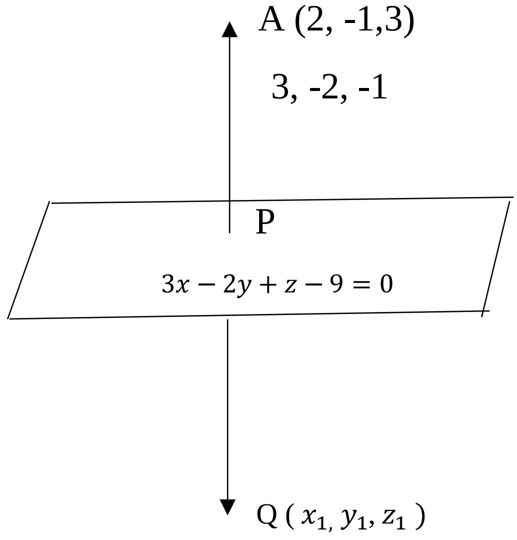
$$\frac{y_1+(-1)}{2} = -3 \Rightarrow y_1 = -6 + 1 = -5$$

And $\frac{z_1+5}{2} = 1 \Rightarrow z_1 = 2 - 5 = -3$

$\therefore$ Image of A is Q = (7, -5, -3)

8.Find the image of the point (2, -1, 3) in the plane $3x - 2y + z - 9 = 0$.

Solution



Let A = (2, -1, 3) and Q = (x_1, y_1, z_1) is the image of A

Direction ratios of AQ = Direction rations of the plane $3x - 2y + z - 9 = 0$

= 3, -2, 1

Equation of the line AQ through A= (2, -1, 3) with d.rs 3, -2, 1

$$\frac{x-2}{3} = \frac{y+1}{-2} = \frac{z-3}{1} = r \text{ say}$$

$$\Rightarrow x = 3r + 2, y = -2r - 1, z = r + 3.$$

Let P = $(3r + 2, -2r - 1, r + 3)$

But the line meets the plane $3x - 2y + z - 9 = 0$

$$\therefore 3(3r + 2) - 2(-2r - 1) + (r + 3) - 9 = 0$$

$$14r + 2 = 0 \Rightarrow r = \frac{-1}{7}$$

$\therefore$ P =$(3(\frac{-1}{7}) + 2, -2(\frac{-1}{7}) - 1, (\frac{-1}{7}) + 3) = (\frac{11}{7}, \frac{-5}{7}, \frac{20}{7})$

Clearly P = The mid-point of A = (2, -1, 3) and Q = (x_1, y_1, z_1)

$$\left(\frac{11}{7}, \frac{-5}{7}, \frac{20}{7}\right) = \left(\frac{x_1+2}{2}, \frac{y_1+(-1)}{2}, \frac{z_1+3}{2}\right)$$

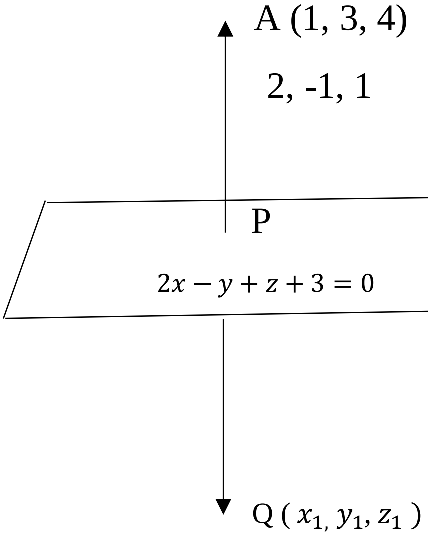
$$\Rightarrow \frac{x_1+2}{2} = \frac{11}{7} \Rightarrow x_1 = \frac{8}{7}$$

$$\frac{y_1+(-1)}{2} = \frac{-5}{7} \Rightarrow y_1 = \frac{-3}{7} \text{ And } \frac{z_1+3}{2} = \frac{20}{7} \Rightarrow z_1 = \frac{19}{7}$$

$\therefore$ Image of A is Q = $(\frac{8}{7}, \frac{-3}{7}, \frac{19}{7})$

9.Find the image of the point (1, 3, 4) in the plane $2x - y + z + 3 = 0.$

Solution



Let A = (1, 3, 4) and Q = (x_1, y_1, z_1) is the image of A

Direction ratios of AQ = Direction rations of the plane $2x - y + z + 3 = 0.$

= 2, -1, 1

Equation of the line AQ through A= (1, 3, 4) with d.rs 2, -1, 1

$$\frac{x-1}{2} = \frac{y-3}{-1} = \frac{z-4}{1} = r\ say$$

$$\Rightarrow x = 2r+1, y = -r+3, z = r+4.$$

Let P = $(2r+1, -r+3, r+4)$

But the line meets the plane $2x - y + z + 3 = 0$

$$\therefore 2(2r+1) - (-r+3) + (r+4) + 3 = 0$$

$$6r + 6 = 0 \Rightarrow r = -1$$

$\therefore$ P =$(2(-1)+1, -(-1)+3, (-1)+4)$ = (-1, 4, 3)

Clearly P = The mid-point of A = (1, 3, 4) and Q = (x_1, y_1, z_1)

$$(-1, 4, 3) = \left(\frac{x_1+1}{2}, \frac{y_1+3}{2}, \frac{z_1+4}{2} \right)$$

$$\Rightarrow \frac{x_1+1}{2} = -1 \Rightarrow x_1 = -2 - 1 = -3$$

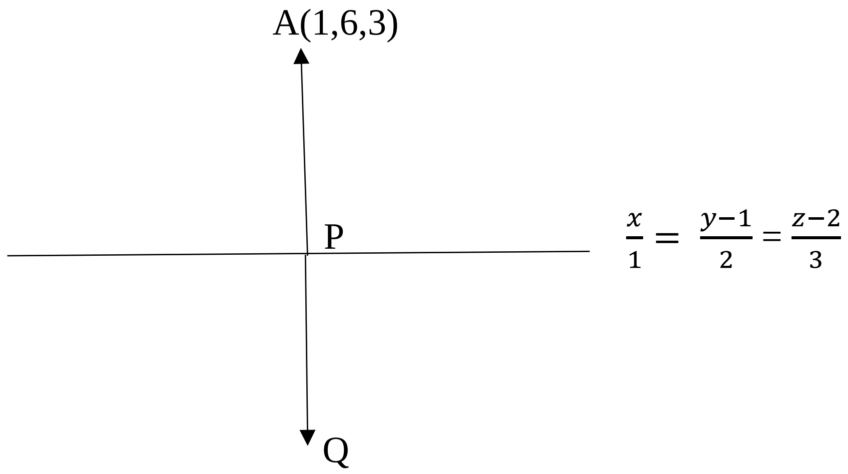
$$\frac{y_1+3}{2} = 4 \Rightarrow y_1 = 8 - 3 = 5$$

And $\frac{z_1+4}{2} = 3 \Rightarrow z_1 = 6 - 4 = 2$

$\therefore$ Image of A is Q = (-3, 5, 2)

10.Find the image of the point (1,6,3) in the line $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$

Solution:



Let A = (1, 6, 3) and Q = (x_1, y_1, z_1) is the image of A

Let $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3} = r\ say \Rightarrow x = r, y = 2r+1, z = 3r+2.$

Let P = $(r, 2r + 1, 3r + 2)$

Drs. Of AP $= r - 1,\ 2r + 1 - 6, 3r + 2 - 3 = r - 1,\ 2r - 5, 3r - 1$

A P is perpendicular to the line $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$

$\therefore 1(r - 1) + 2(2r - 5) + 3(3r - 1) = 0 \Rightarrow 14r - 14 = 0 \Rightarrow r = 1$

$\therefore$ P = $(r, 2r + 1, 3r + 2) = (1, 2(1) + 1, 3(1) + 2) = (1, 3, 5)$

But P $= (1, 3, 5)$ is the midpoint of A = (1, 6, 3) and Q = (x_1, y_1, z_1)

$$= \left(\frac{x_1+1}{2}, \frac{y_1+6}{2}, \frac{z_1+3}{2} \right)$$

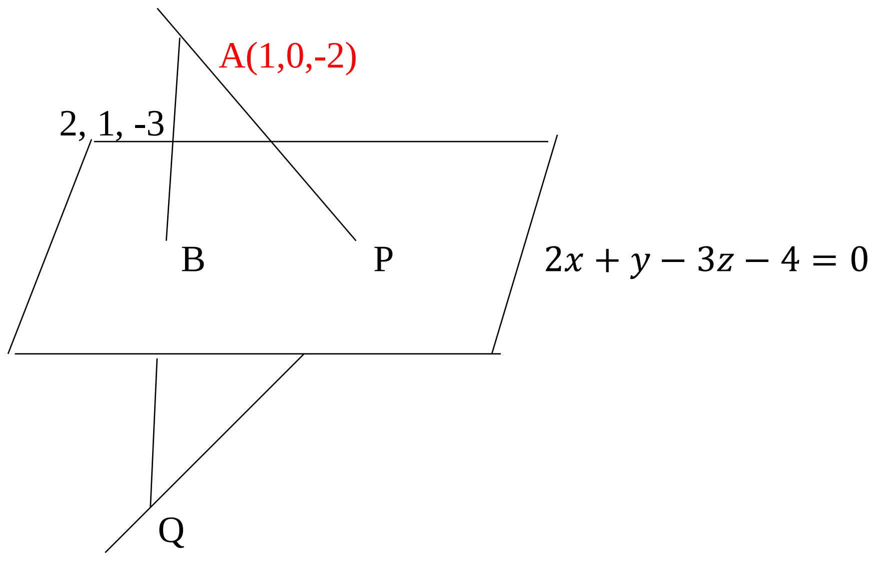
$\frac{x_1+1}{2} = 1 \Rightarrow x_1 = 2 - 1 = 1$

$\frac{y_1+6}{2} = 3 \Rightarrow y_1 = 6 - 6 = 0$

$\frac{z_1+3}{2} = 5 \Rightarrow z_1 = 10 - 3 = 7$ $\therefore$ Image of A is Q = (1, 0, 7).

11.Find the image of the line $\frac{x-1}{2} = \frac{y}{-1} = \frac{z+2}{2}$ in the plane $2x + y - 3z - 4 = 0.$

Solution:



Let $\frac{x-1}{2} = \frac{y}{-1} = \frac{z+2}{2} = r \Rightarrow x = 2r + 1, y = -r, z = 2r - 2$

$$P = (2r + 1, -r, 2r - 2)$$

But the line intersects the plane $2x + y - 3z - 4 = 0$

$$\therefore\ 2(2r + 1) + (-r) - 3(2r - 2) - 4 = 0$$

$$4r + 2 - r - 6r + 6 - 4 = 0 \Rightarrow -3r = -4 \Rightarrow r = \frac{4}{3}$$

$$\therefore\ P = (2r+1, -r, 2r-2) = (\ 2(\tfrac{4}{3}) + 1,\ -(\tfrac{4}{3}),\ 2(\tfrac{4}{3}) - 2\ )$$

P $= (\frac{11}{3}, \frac{-4}{3}, \frac{2}{3})$

In the given line $\frac{x-1}{2} = \frac{y}{-1} = \frac{z+2}{2}$ clearly the point A = (1, 0, -2)

Now to find the image of the point A = (1, 0, -2) in the plane

$$2x + y - 3z - 4 = 0$$

Let it be Q = (x_1, y_1, z_1)

Direction ratios of AQ = Direction rations of the plane $2x + y - 3z - 4 = 0$.

= 2, 1, -3

Equation of the line AQ through A= (1, 0, -2) with d.rs 2, 1, -3

$$\frac{x-1}{2} = \frac{y-0}{1} = \frac{z+2}{-3} = r\ say$$

$$\Rightarrow x = 2r+1, y = r, z = -3r - 2.$$

Let P = $(2r+1, r, -3\,r - 2)$

But the line meets the plane $2x + y - 3z - 4 = 0$

$$\therefore 2(2r+1) + (r) - 3(-3r-2) - 4 = 0$$

$$14r + 4 = 0 \Rightarrow r = -\frac{2}{7}$$

$$\therefore \text{P} = (2\left(-\frac{2}{7}\right) + 1, -\frac{2}{7}, -3\,(-\frac{2}{7}) - 2) = (\frac{3}{7}, \frac{-2}{7}, \frac{-8}{7})$$

Clearly P = The mid-point of A = (1, 0, -2) and Q = (x_1, y_1, z_1)

P $(\frac{3}{7}, \frac{-2}{7}, \frac{-8}{7}) = (\frac{x_1+1}{2}, \frac{y_1+0}{2}, \frac{z_1-2}{2})$

$$\Rightarrow \frac{x_1+1}{2} = \frac{3}{7} \Rightarrow x_1 = \frac{6}{7} - 1 = -\frac{1}{7}$$

$$\frac{y_1+0}{2} = \frac{-2}{7} \Rightarrow y_1 = \frac{-4}{7}$$

And $\frac{z_1-2}{2} = \frac{-8}{7} \Rightarrow z_1 = \frac{-16}{7} + 2 = \frac{-2}{7}$

$\therefore$ Image of A = (1, 0, -2) is **Q = $\left(-\frac{1}{7}, \frac{-4}{7}, \frac{-2}{7} \right)$**

Direction ratios of the line P $= \left(\frac{11}{3}, \frac{-4}{3}, \frac{2}{3} \right)$, Q = $(-\frac{1}{7}, \frac{-4}{7}, \frac{-2}{7})$

are $\frac{-1}{7}-\frac{11}{3}, \frac{-4}{7}+\frac{4}{3}, \frac{-2}{7}-\frac{2}{3} = \frac{-80}{21}, \frac{16}{21}, \frac{-20}{21}$

Equation of the image line through the point Q = $\left(-\frac{1}{7}, \frac{-4}{7}, \frac{-2}{7}\right)$

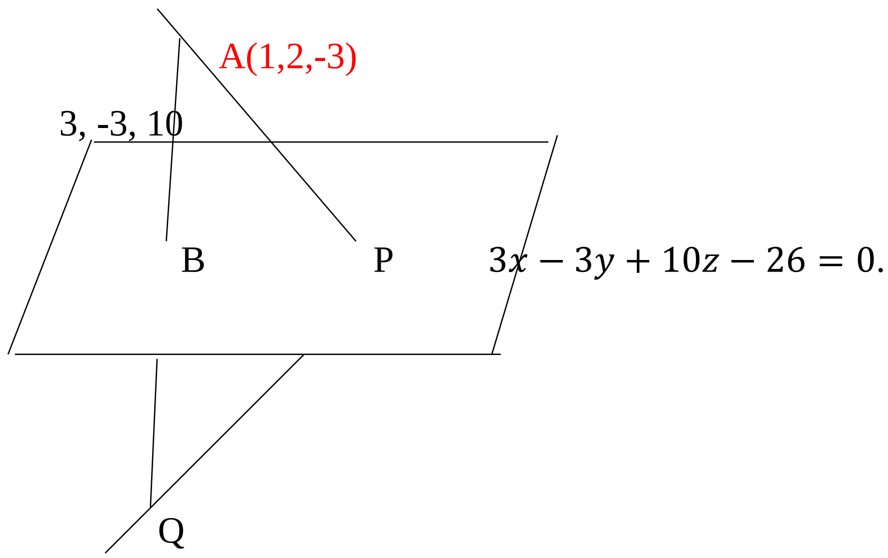
$$\frac{x+\frac{1}{7}}{\frac{-80}{21}} = \frac{y+\frac{4}{7}}{\frac{16}{21}} = \frac{z+\frac{2}{7}}{\frac{-20}{21}} \Rightarrow \frac{x+\frac{1}{7}}{-80} = \frac{y+\frac{4}{7}}{16} = \frac{z+\frac{2}{7}}{-20}$$

$$\frac{x+\frac{1}{7}}{20} = \frac{y+\frac{4}{7}}{-4} = \frac{z+\frac{2}{7}}{5}$$

12.Find the image of the line $\frac{x-1}{9} = \frac{y-2}{1} = \frac{z+3}{-3}$ in the plane

$3x - 3y + 10z - 26 = 0.$

Solution:



$Let\ \frac{x-1}{9} = \frac{y-2}{1} = \frac{z+3}{-3} = r \Rightarrow x = 9r+1, y = r+2, z = -3r-3$

$$P = (9r+1, r+2, -3r-3)$$

But the line intersects the plane $3x - 3y + 10z - 26 = 0.$

$$\therefore\ 3(9r+1) - 3(r+2) + 10(-3r-3) - 26 = 0.$$

$$27r + 3 - 3r - 6 - 30r - 30 - 26 = 0 \Rightarrow -6r = 59 \Rightarrow r = -\frac{59}{6}$$

$$\therefore\ P = (9r+1, r+2, -3r-3)$$

$$= \left(9\left(-\frac{59}{6}\right)+1, -\frac{59}{6}+2, -3\left(-\frac{59}{6}\right) - 3\right)$$

$P = \left(\frac{-525}{6}, \frac{-47}{6}, \frac{159}{6} \right)$

In the given line $\frac{x-1}{9} = \frac{y-2}{1} = \frac{z+3}{-3}$ clearly the point A = (1, 2, -3)

Now to find the image of the point A = (1, 2, -3) in the plane

$3x - 3y + 10z - 26 = 0.$

Let it be Q = (x_1, y_1, z_1)

Direction ratios of AQ = Direction rations of the plane. $3x - 3y + 10z - 26 = 0.$

= 3, -3, 10

Equation of the line AQ through A= (1, 2, -3) with d.rs 3, -3, 10

$$\frac{x-1}{3} = \frac{y-2}{-3} = \frac{z+3}{10} = r \ say$$

$$\Rightarrow x = 3r + 1, y = -3r + 2, z = 10r - 3.$$

Let P = ($3r + 1, -3r + 2, 10\,r - 3$)

But the line meets the plane $3x - 3y + 10z - 26 = 0.$

$$\therefore 3(3r + 1) - 3(-3r + 2) + 10(10r - 3) - 26 = 0.$$

$$118r - 59 = 0 \Rightarrow r = \frac{1}{2}$$

$\therefore$ P = ($3r + 1, -3r + 2, 10\,r - 3$)

$$\left(3(\tfrac{1}{2}) + 1, -3(\tfrac{1}{2}) + 2, 10\,(\tfrac{1}{2}) - 3 \right) = \left(\tfrac{5}{2}, \tfrac{1}{2}, 2 \right)$$

Clearly P = The mid-point of A = (1, 2, -3) and Q = (x_1, y_1, z_1)

$$P\left(\tfrac{5}{2}, \tfrac{1}{2}, 2 \right) = \left(\frac{x_1+1}{2}, \frac{y_1+2}{2}, \frac{z_1-3}{2} \right)$$

$$\Rightarrow \frac{x_1+1}{2} = \frac{5}{2} \Rightarrow x_1 = \frac{10}{2} - 1 = \frac{8}{2} = 4$$

$$\frac{y_1+2}{2} = \frac{1}{2} \Rightarrow y_1 = -1$$

And $\frac{z_1-3}{2} = 2 \Rightarrow z_1 = 7$ $\quad\therefore$ Image of A is Q = (4, -1, 7)

Direction ratios of the line $P = \left(\frac{-525}{6}, \frac{-47}{6}, \frac{159}{6} \right)$, Q = (4, -1, 7)

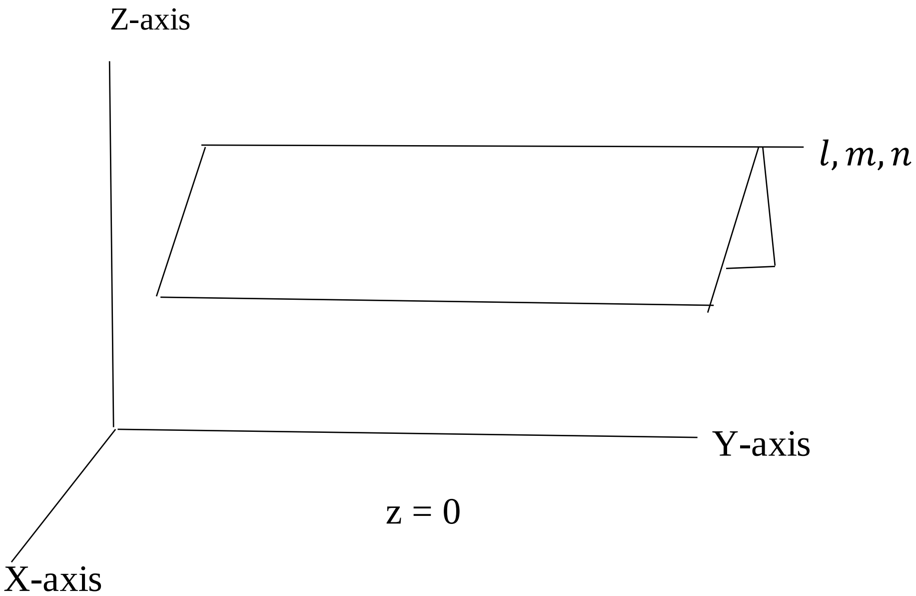
are 549/6, 41/6, -117/6

Equation of the image line through the point Q = ($4, -1, 7$)

$$\frac{x-4}{549} = \frac{y+1}{41} = \frac{z-7}{-117}$$

13.Find the symmetric form of the line $2x + 2y - z - 6 = 0 = 2x + 3y - z - 8$

Solution:



Let l, m, n *are drs of the line* $2x + 2y - z - 6 = 0 = 2x + 3y - z - 8$

We know that the drs of the plane are normal to the plane and drs of the line lies on the line.

$\therefore\ l(2) + m(2) + n(-1) = 0$ --------(1)

And $l(2) + m(3) + n(-1) = 0$ -------(2)

To solve (1)& (2) 2 -1 2 2

3 -1 2 3

$\frac{l}{-2+3} = \frac{m}{-2+2} = \frac{n}{6-4} \Rightarrow \frac{l}{1} = \frac{m}{0} = \frac{n}{2}$

Let us project the line in XY-Plane i.e. z = 0

The given line can be written as

$$2x + 2y - 6 = 0$$

$$2x + 3y - 8 = 0$$

To solve the equations 2 -6 2 2

3 -8 2 3

$\frac{x}{-16+18} = \frac{y}{-12+18} = \frac{1}{6-4} \Rightarrow \frac{x}{2} = \frac{y}{6} = \frac{1}{2} \Rightarrow x = 1, y = 3\ \&\ z = 0$

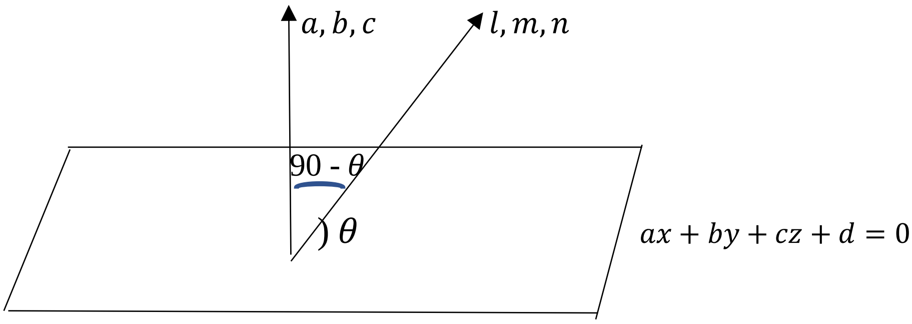
$\therefore$ The symmetrical form of the line through (1, 3, 0) d.r's 1, 0, 2 is

$$\frac{x-1}{1} = \frac{y-3}{0} = \frac{z-0}{2}$$

Angle between a line and a plane:

1.If θ is the angle between a plane $ax + by + cz + d = 0$and the line $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$ then

$$sin\theta = \frac{al+bm+cn}{\sqrt{a^2+b^2+c^2}\sqrt{l^2+m^2+n^2}}$$



2.A line and a plane are parallel to each other if $al + bm + cn = 0$

and perpendicular if $\frac{a}{l} = \frac{b}{m} = \frac{c}{n}$

3. The line $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$ lies in the plane $ax + by + cz + d = 0$

if and only if (i) $ax_1 + by_1 + cz_1 + d = 0$ and (ii) $al + bm + cn = 0$.

Problems:

1.Show that a line $\frac{x-2}{1} = \frac{y-2}{2} = \frac{z+1}{3}$ lies in the plane $5x + 2y - 3z - 17 = 0$

Solution: Now (x_1, y_1, z_1) = (2, 2, -1) and $l, m, n = 1, 2, 3$

$$ax_1 + by_1 + cz_1 + d = 5(2) + 2(2) - 3(-1) - 17 = 10 + 4 + 3 - 17 = 0$$

And $al + bm + cn = 5(1) + 2(2) - 3(3) = 5 + 4 - 9 = 0$

∴ The line lies on the plane

2.Show that a line $\frac{x+1}{-1} = \frac{y+2}{3} = \frac{z+5}{5}$ lies in the plane $x + 2y - z = 0$

Solution: Now (x_1, y_1, z_1) = (-1, -2, -5) and $l, m, n = -1, 3, 5$

$$ax_1 + by_1 + cz_1 + d = (-1) + 2(-2) - (-5) = -1 - 4 + 5 = 0$$

And $al + bm + cn = 1(-1) + 2(3) - (5) = -1 + 6 - 5 = 0$

∴ The line lies on the plane.

3.Find the angle between the line $\frac{x+1}{2} = \frac{y}{3} = \frac{z-3}{6}$ and the plane $3x + y + z = 7$.

Solution: $l, m, n = 2, 3, 6$ $and\ a, b,\ c = 3, 1, 1$

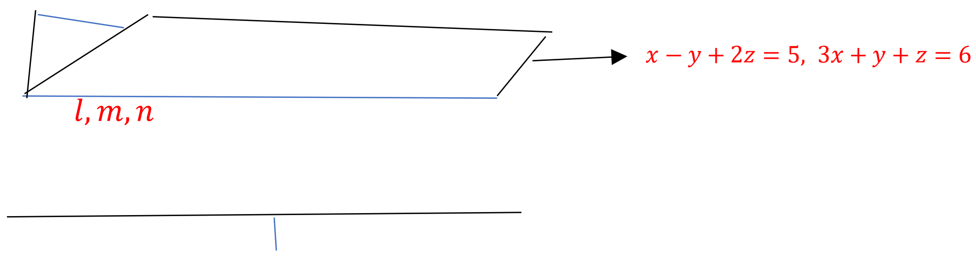
Now $sin\theta = \frac{al+bm+cn}{\sqrt{a^2+b^2+c^2}\sqrt{l^2+m^2+n^2}} = \frac{2(3)+3(1)+6(1)}{\sqrt{2^2+3^2+6^2}\sqrt{3^2+1^2+1^2}} = \frac{15}{\sqrt{49}\sqrt{11}}$

$$\sin\theta = \frac{15}{7\sqrt{11}} \Rightarrow \theta = \sin^{-1}\frac{15}{7\sqrt{11}}$$

4. Find the equation of a line through the point (1, 2, 3) and parallel to a line

$$x - y + 2z = 5,\ 3x + y + z = 6$$

Solution:



A(1, 2, 3)

Let l, m, n are direction ratios of the line $x - y + 2z - 5 = 0 = 3x + y + z - 6$

We know the Drs of the plane are normal to the plane

$\therefore$ $l(1) + m(-1) + n(2) = 0$ --------(1)

And $l(3) + m(1) + n(1) = 0$ ---------(2)

To solve (1) and (2)

-1	2	1	-1
1	1	3	1

$$\frac{l}{-1-2} = \frac{m}{6-1} = \frac{n}{1-(-1)} \Rightarrow \frac{l}{-3} = \frac{m}{5} = \frac{n}{2}$$

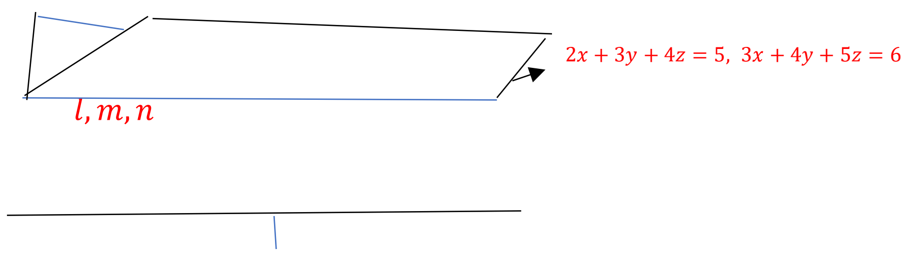
$\therefore$ *Equation of the line through the point* (1,2,3) *with drs* $-3,5,2$

$$\frac{x-1}{-3} = \frac{y-2}{5} = \frac{z-3}{2}$$

5. Find the equation of a line through the point (-2, 3, 4) and parallel to a line

$$2x + 3y + 4z = 5,\ 3x + 4y + 5z = 6$$

Solution:



A(-2,3,4)

Let l, m, n are direction ratios of the line $2x + 3y + 4z = 5,\ 3x + 4y + 5z = 6$

We know the Drs of the plane are normal to the plane

$$\therefore \quad l(2) + m(3) + n(4) = 0 \text{ --------(1)}$$

$$\text{And } \ l(3) + m(4) + n(5) = 0 \text{ ---------(2)}$$

To solve (1) and (2)

3	4	2	3
4	5	3	4

$$\frac{l}{15-16} = \frac{m}{12-10} = \frac{n}{8-9} \Rightarrow \frac{l}{-1} = \frac{m}{2} = \frac{n}{-1}$$

$\therefore$ *Equation of the line through the point* $(1,2,3)$ *with drs* $-3,5,2$

$$\frac{x-1}{1} = \frac{y-2}{-2} = \frac{z-3}{1}$$

6.Find the equation of the plane containing the line

$x - 3y + 2z + 3 = 0 = \ 3x - y + 2z - 5$ and through the origin.

Solution:

Equation of the plane containing the line $x - 3y + 2z + 3 = 0 = \ 3x - y + 2z - 5$

= Equation of the plane passes through the intersection of planes

$$\pi_1 = x - 3y + 2z + 3 = 0$$

$\pi_2 = 3x - y + 2z - 5 = 0$ and it is $\pi_1 + \lambda\ \pi_2 = 0$

$$\therefore \quad x - 3y + 2z + 3 + \lambda(3x - y + 2z - 5) = 0 \text{ ------(1)}$$

But it is passes through origin (0, 0, 0)

$$\therefore \quad 0 - 0 + 0 + 3 + \lambda(0 - 0 + 0 - 5) = 0 \Rightarrow \lambda = \frac{3}{5}$$

Equation (1) can be written as

$$x - 3y + 2z + 3 + \tfrac{3}{5}(3x - y + 2z - 5) = 0$$

$$\Rightarrow 14x - 18y + 16z = 0 \ i.e, 7x - 9y + 8z = 0$$

7.Find the equation of the plane containing the line

$x - y + 3z + 5 = 0 = \ 2x + y - 2z + 6$ and through the point (3, 1, 1).

Solution:

Equation of the plane containing the line $x - y + 3z + 5 = 0 = \ 2x + y - 2z + 6$

= Equation of the plane passes through the intersection of planes

$$\pi_1 = x - y + 3z + 5 = 0$$

$$\pi_2 = 2x + y - 2z + 6 = 0 \text{ and it is } \pi_1 + \lambda\, \pi_2 = 0$$

$$\therefore\; x - y + 3z + 5 + \lambda(2x + y - 2z + 6) = 0 \text{ ------(1)}$$

But it is passes through origin (3, 1, 1)

$$\therefore\; 3 - 1 + 3(1) + 5 + \lambda(2(3) + 1 - 2(1) + 6) = 0$$

$$\Rightarrow 10 + \lambda(11) \;\Rightarrow\; \lambda = \frac{-10}{11}$$

Equation (1) can be written as

$$x - y + 3z + 5 + \frac{-10}{11}(2x + y - 2z + 6) = 0$$

$$\Rightarrow -9x - 21y + 53z - 5 = 0$$

8.Find the equation of the plane containing the parallel lines

$$\frac{x-4}{1} = \frac{y-3}{-4} = \frac{z-2}{5}\;;\; \frac{x-3}{1} = \frac{y+2}{-4} = \frac{z}{5}$$

Solution: In the given lines

$$\frac{x-4}{1} = \frac{y-3}{-4} = \frac{z-2}{5}\;;\; \frac{x-3}{1} = \frac{y+2}{-4} = \frac{z}{5}$$

Clearly A= (4, 3, 2) B = (3, -2, 0) are the points and the drs are 1, -4, 5

Let a, b, c are direction ratios of the normal line to the required plane.

We know that Equation of a plane passing through P ($x_{1,}y_1,z_1$) and whose direction ratios of the normal line are a, b, c is

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0.$$

∴ Equation passes through the point A = (4, 3, 2) is

$$a(x - 4) + b(y - 3) + c(z - 2) = 0 \text{ ----------(1)}$$

But it passes through the point B = (3, -2, 0)

$$\therefore\; a(3 - 4) + b(-2 - 3) + c(0 - 2) = 0$$

$$a(-1) + b(-5) + c(-2) = 0$$

$$a(1) + b(5) + c(2) = 0 \quad \text{--------------(2)}$$

Also, it is parallel line

$$\frac{x-4}{1} = \frac{y-3}{-4} = \frac{z-2}{5} \text{ Condition } al + bm + cn = 0$$

$$\therefore\ a(1) + b(-4) + c(5) = 0 \text{ ---------------(3)}$$

To solve the Equations (2) and (3) using the method of cross multiplication

5	2	1	5
-4	5	1	-4

$$\frac{a}{25+8} = \frac{b}{2-5} = \frac{c}{-4-5}$$

$$\frac{a}{33} = \frac{b}{-3} = \frac{c}{-9} \quad \Rightarrow \quad \frac{a}{11} = \frac{b}{-1} = \frac{c}{-3}$$

Put the values in (1)

$$11(x-4) - 1(y-3) - 3(z-2) = 0$$

$11x - y - 3z - 35 = 0$ is required plane

9.Find the equation of the plane containing the parallel lines

$$\frac{x-3}{4} = \frac{y-2}{-5} = \frac{z-4}{-1}\ ;\ \frac{x+2}{-4} = \frac{y}{5} = \frac{z-3}{1}$$

Solution: In the given lines

$$\frac{x-3}{4} = \frac{y-2}{-5} = \frac{z-4}{-1}\ ;\ \frac{x+2}{-4} = \frac{y}{5} = \frac{z-3}{1}$$

Clearly A= (3, 2, 4) B = (-2, 0, 3) are the points and the drs are 4, -5,-1

Let a, b, c are direction ratios of the normal line to the required plane.

We know that Equation of a plane passing through P (x_1, y_1, z_1) and whose direction ratios of the normal line are a, b, c is

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0.$$

$\therefore$ Equation passes through the point A = (3, 2, 4) is

$$a(x-3) + b(y-2) + c(z-4) = 0 \text{ ----------(1)}$$

But it passes through the point B = (-2, 0, 3)

$$a(-2-3) + b(0-2) + c(3-4) = 0$$

$$a(-5) + b(-2) + c(-1) = 0$$

$$a(5) + b(2) + c(1) = 0 \qquad \text{--------------(2)}$$

Also, it is parallel line $\frac{x-3}{4} = \frac{y-2}{-5} = \frac{z-4}{-1}$

Condition $al + bm + cn = 0$

$$\therefore\ a(4) + b(-5) + c(-1) = 0 \text{ ---------------(3)}$$

To solve the Equations (2) and (3) using the method of cross multiplication

2	1	5	2
-5	-1	4	-5

$$\frac{a}{-2+5} = \frac{b}{4+5} = \frac{c}{-25-8}$$

$$\frac{a}{3} = \frac{b}{9} = \frac{c}{-33} \quad \Rightarrow \quad \frac{a}{1} = \frac{b}{3} = \frac{c}{-11}$$

Put the values in (1)

$$1(x-3) + 3(y-2) - 11(z-4) = 0$$

$$\Rightarrow x + 3y - 11z + 35 = 0$$ is required plane

10.Find the equation of the plane through the points (2, -1,0) (3, -4, 5) and parallel to $3x = 2y = z$.

Solution: Let a, b, c are direction ratios of the normal line to the plane.

We know that Equation of a plane passing through P (x_1, y_1, z_1) and whose direction ratios of the normal line are a, b, c is

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0.$$

$\therefore$ Equation passes through the point (2, 2, 1) is

$$a(x-2) + b(y+1) + c(z-0) = 0 \text{ ----------(1)}$$

But it passes through the point (3, -4, 5))

$$\therefore\ a(3-2) + b(-4+1) + c(5-0) = 0$$

$$a(1) + b(-3) + c(5) = 0 \qquad \text{--------------(2)}$$

Also, it is parallel to the line $3x = 2y = z \Rightarrow \frac{x}{2} = \frac{z}{3} = \frac{z}{6}$

$$\therefore\ a(2) + b(3) + c(6) = 0 \text{ ---------------(3)}$$

To solve the Equations (2) and (3) using the method of cross multiplication

-3	5	1	-3
3	6	2	3

$$\frac{a}{-18-15} = \frac{b}{10-6} = \frac{c}{3+6}$$

$$\frac{a}{-33} = \frac{b}{4} = \frac{c}{9}$$

Put the values in (1) $-33(x-2)+4(y+1)+9(z-0)=0$

$-33x+4y+9z+70=0$ is required plane

11.Find the equation of the plane through the points (1, 0, -1) (3,2, 2) and parallel to the line $\frac{x-1}{1}=\frac{y-1}{-2}=\frac{z-2}{3}$

Solution: Let a, b, c are direction ratios of the normal line to the plane.

We know that Equation of a plane passing through P (x_1, y_1, z_1) and whose direction ratios of the normal line are a, b, c is

$$a(x-x_1)+b(y-y_1)+c(z-z_1)=0.$$

∴ Equation passes through the point (1, 0, -1) is

$$a(x-1)+b(y-0)+c(z+1)=0 \text{ ----------(1)}$$

But it passes through the point (3, 2, 2)

$$\therefore\ a(3-1)+b(2-0)+c(2+1)=0$$

$$a(2)+b(2)+c(3)=0 \qquad \text{--------------(2)}$$

Also, it is parallel to the line $\frac{x-1}{1}=\frac{y-1}{-2}=\frac{z-2}{3}$

$$\therefore\ a(1)+b(-2)+c(3)=0 \text{ ---------------(3)}$$

To solve the Equations (2) and (3) using the method of cross multiplication

2	3	2	2
-2	3	1	-2

$$\frac{a}{6+6}=\frac{b}{3-6}=\frac{c}{-4-2}$$

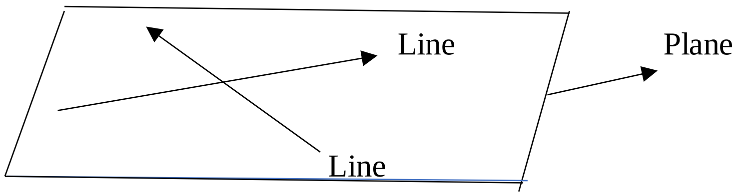
$$\frac{a}{12}=\frac{b}{-3}=\frac{c}{-6} \Rightarrow \frac{a}{4}=\frac{b}{-1}=\frac{c}{-2}$$

Put the values in (1) $4(x-1)-1(y-0)-2(z+1)=0$

$\Rightarrow 4x-y-2z-6=0$ is required plane.

Coplanar lines:

The given two lines lie in the same plane then we say the lines are coplanar.



Coplanar lines

Condition for the two lines

$\frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1}$ and $\frac{x-x_2}{l_2} = \frac{y-y_2}{m_2} = \frac{z-z_2}{n_2}$ are coplanar

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

And the equation of the plane containing them is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

Problems:

1.Show that lines $\frac{x+3}{-4} = \frac{y+1}{7} = \frac{z-4}{1}$; $\frac{x+1}{-3} = \frac{y+10}{8} = \frac{z-1}{2}$

are coplanar and find the equation of the plane cotaining them.

Solution: In the given problem points on the lines

A= ($x_1,\ y_1,\ z_1$) = (-3,-1, 4) B = ($x_2,\ y_2,\ z_2$) = (-1, -10, 1)

and the d.rs of the lines

$$l_1, m_1, n_1 = -4, 7, 1 \quad and \quad l_2, m_2, n_2 = -3, 8, 2$$

Now condition for coplanar lines

LHS = $\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = \begin{vmatrix} -1+3 & -10+1 & 1-4 \\ -4 & 7 & 1 \\ -3 & 8 & 2 \end{vmatrix}$

= $\begin{vmatrix} 2 & -9 & -3 \\ -4 & 7 & 1 \\ -3 & 8 & 2 \end{vmatrix}$ = 2(14 - 8) + 9(-8 + 3) -3(-32 +21)

= 12 - 45 + 33 = 0 = RHS

∴ The lines are coplanar

Equation of the plane

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

$$\begin{vmatrix} x+3 & y+1 & z-4 \\ -4 & 7 & 1 \\ -3 & 8 & 2 \end{vmatrix} = 0$$

$$\Rightarrow (x+3)(14-8) - (y+1)(-8+3) + (z-4)(-32+21) = 0$$

$$\Rightarrow 6(x+3) + 5(y+1) - 11(z-4) = 0$$

$$\Rightarrow 6x + 5y - 11z + 67 = 0$$

2.Show that lines $\frac{x+2}{3} = \frac{y-1}{2} = \frac{z-2}{-1}$; $\frac{x-3}{-2} = \frac{y-2}{1} = \frac{z+1}{2}$ *are coplanar and find the equation of the plane cotaining them.*

Solution: In the given problem points on the lines

A= (x_1, y_1, z_1) = (-2,1, 2) B = (x_2, y_2, z_2) = (3, 2, -1)

and the d.rs of the lines

$$l_1, m_1, n_1 = 3, 2, -1 \quad and \quad l_2, m_2, n_2 = -2, 1, 2$$

Now condition for coplanar lines

LHS = $\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} =$

$= \begin{vmatrix} 5 & 1 & -3 \\ 3 & 2 & -1 \\ -2 & 1 & 2 \end{vmatrix}$ = 5(4 + 1) -1(6 - 2) - 3(3 + 4)

= 25- 4- 21 = 0 = RHS

∴ The lines are coplanar

Equation of the plane

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

$$\begin{vmatrix} x + 2 & y - 1 & z - 2 \\ 3 & 2 & -1 \\ -2 & 1 & 2 \end{vmatrix} = 0$$

$$\Rightarrow (x+2)(4+1) - (y-1)(6-2) + (z-2)(3+4) = 0$$

$$\Rightarrow 5(x+2) - 4(y-1) + 7(z-2) = 0$$

$$\Rightarrow \quad 5x - 4y + 7z = 0.$$

3.Show that the lines

$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$; $\frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5}$ are coplanar and find the points of intersection and the plane containing them.

Solution: Let $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4} = r \Rightarrow x = 2r+1, y = 3r+2, z = 4r+3$

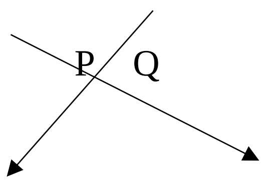
$$Let\ \ P = (2r+1, 3r+2, 4r+3)$$

$$and\ \frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5} = r' \Rightarrow x = 3r'+2, y = 4r'+3, z = 5r'+4$$

$$Let\ Q = (3r'+2, 4r'+3, 5r'+4)$$

If the lines are intersected then P = Q

$\Rightarrow (2r+1, 3r+2, 4r+3) = (3r'+2, 4r'+3, 5r'+4)$



$2r + 1 = 3r' + 2 \Rightarrow \ 2r - 3r' - 1 = 0$ -----------(1)

$$3r+2=,4r'+3 \Rightarrow 3r-4r'-1=0 \text{ ----------(2)}$$

$$4r+3=5r'+4 \Rightarrow 4r-5r'-1=0 \text{ ----------(3)}$$

To solve (1) & (2)

-3	-1	2	-3
-4	-1	3	-4

$$\frac{r}{3-4}=\frac{r\prime}{-3+2}=\frac{1}{-8+9} \Rightarrow \frac{r}{-1}=\frac{r\prime}{-1}=\frac{1}{1} \Rightarrow r=-1 \text{ and } r'=-1$$

Put the values in (3) LHS = $4r-5r'-1=4(-1)-5(-1)-1=0$

∴ The lines are intersection

Point of intersection

$P=(2r+1,3r+2,4r+3)=(2(-1)+1,3(-1)+2,4(-1)+3)=$ **(-1, -1, -1)**

Equation of the plane

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0$$

$$\Rightarrow \begin{vmatrix} x-1 & y-2 & z-3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix}=0$$

$$\Rightarrow (x-1)(15-16)-(y-2)(10-12)+(z-3)(8-9)=0$$

$$\Rightarrow -1(x-1)+2(y-2)-1(z-3)=0$$

$$\Rightarrow -x+2y-z=0 \Rightarrow x-2y+z=0$$

4.Show that the lines

$\frac{x-1}{2}=\frac{y+1}{-3}=\frac{z+10}{8}$; $\frac{x-4}{1}=\frac{y+3}{-4}=\frac{z+1}{7}$ are coplanar and find the points of intersection and the plane containing them.

Solution:

Let $\frac{x-1}{2}=\frac{y+1}{-3}=\frac{z+10}{8}=r \Rightarrow x=2r+1, y=-3r-1, z=8r-10$

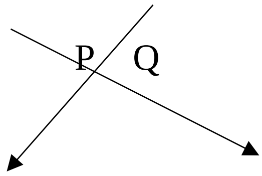
$$Let\ \ P=(2r+1,-3r-1,8r-10)$$

$$and\ \frac{x-4}{1}=\frac{y+3}{-4}=\frac{z+1}{7}=r' \Rightarrow x=r'+4, y=-4r'-3, z=7r'-1$$

$$Let\ Q=(r'+4,-4r'-3,7r'-1)$$

If the lines are intersected then P = Q

$\Rightarrow (\,2r+1, -3r-1, 8r-10) \;= (r'+4, -4r'-3, 7r'-1)$



$$2r+1 = r'+4 \Rightarrow \;\; 2r - r' - 3 = 0 \text{ -----------(1)}$$

$$-3r-1 = \; -4r' - 3 \Rightarrow -3r + 4r' + 2 = 0 \;\text{ ----------(2)}$$

$$8r - 10 = 7r' - 1 \Rightarrow 8r - 7r' - 9 = 0 \;\text{ ----------(3)}$$

To solve (1) & (2)

-1	-3	2	-1
4	2	-3	4

$$\frac{r}{-2+12} = \frac{r'}{9-4} = \frac{1}{8-3} \Rightarrow \frac{r}{10} = \frac{r'}{5} = \frac{1}{5} \Rightarrow r = 2 \; and \; r' = 1$$

Put the values in (3) LHS = $8r - 7r' - 9 = 8(2) - 7(1) - 9 = 0$=RHS

∴ The lines are intersection

Point of intersection

$Q\; = (r'+4, -4r'-3, 7r'-1) = Q\; = (1+4, -4(1)-3, 7(1)-1)$= **(5, -7, 6)**

Equation of the plane

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x-4 & y+3 & z+1 \\ 2 & -3 & 8 \\ 1 & -4 & 7 \end{vmatrix} = 0$$

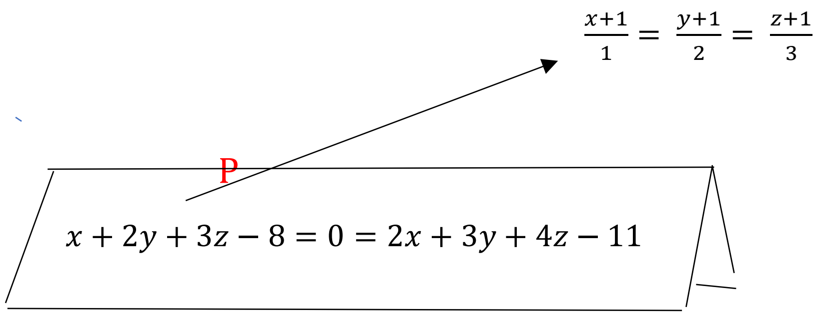
$\Rightarrow (x-4)(-21+32) - (y+3)(14-8) + (z+1)(-8+3) = 0$

$\Rightarrow 11(x-4) - 6(y+3) - 5(z+1) = 0$

$\Rightarrow \mathbf{11x - 6y - 5z - 67 = 0}$

5.Show that the lines $\frac{x+1}{1} = \frac{y+1}{2} = \frac{z+1}{3}$ and $x + 2y + 3z - 8 = 0 = 2x + 3y + 4z - 11$ are intersecting. Find the point of contact and the plane containing them.

Solution:



Let $\frac{x+1}{1} = \frac{y+1}{2} = \frac{z+1}{3} = r \Rightarrow x = r - 1, y = 2r - 1, z = 3r - 1$

And let P = $(r - 1, 2r - 1, 3r - 1)$

If the lines are intersecting P lies on the line $x + 2y + 3z - 8 = 0 = 2x + 3y + 4z - 11$

$$x + 2y + 3z - 8 = 0 \Rightarrow (r - 1) + 2(2r - 1) + 3(3r - 1) - 8 = 0$$

$$14r - 14 = 0 \Rightarrow r = 1$$

Now P = $(r - 1, 2r - 1, 3r - 1)$ = $(1 - 1, 2 - 1, 3 - 1)$ = (0,1, 2)

Put this point in the second plane $2x + 3y + 4z - 11 = 0$

LHS = $2x + 3y + 4z - 11 = 2(0) + 3(1) + 4(2) - 11 = 0 = RHS$

∴ The lines are intersecting at P (0,1, 2)

Equation of the plane containing the line $x + 2y + 3z - 8 = 0 = 2x + 3y + 4z - 11$ is

$$x + 2y + 3z - 8 + \lambda(2x + 3y + 4z - 11) = 0 \text{ ---------(1)}$$

$$(1 + 2\lambda)x + (2 + 3\lambda)y + 3 + 4\lambda)z - (8 + 11\lambda) = 0$$

But the line $\frac{x+1}{1} = \frac{y+1}{2} = \frac{z+1}{3}$ $lies\ on\ the\ plane$

$$\therefore (1 + 2\lambda)(1) + (2 + 3\lambda)(2) + (3 + 4\lambda)(3) = 0$$

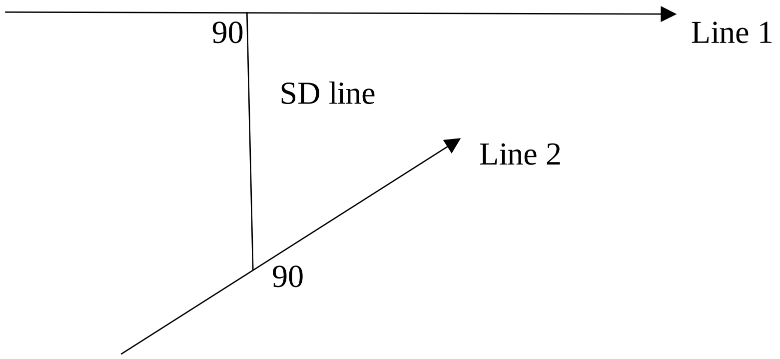
$$14 + 20\lambda = 0 \Rightarrow \lambda = \frac{-7}{10}$$ put the value in (1)

$$x + 2y + 3z - 8 + \frac{-7}{10}(2x + 3y + 4z - 11) = 0$$

$$-4x - y + 2y - 3 = 0 \Rightarrow 4x + y - 2y + 3 = 0$$

Shortest Distance:

Shortest distance is the least distance between the two lines and it is the perpendicular line to given lines.



1.The Length of the shortest distance between the lines

$$\frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1} \text{ and } \frac{x-x_2}{l_2} = \frac{y-y_2}{m_2} = \frac{z-z_2}{n_2} \text{ is}$$

$$l(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)$$

where l, m, n *are direction cosines of the SD line.*

And Equation of SD line is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l & m & n \end{vmatrix} = 0 = \begin{vmatrix} x-x_2 & y-y_2 & z-z_2 \\ l_2 & m_2 & n_2 \\ l & m & n \end{vmatrix} = 0$$

2.If the shortest distance of the lines is 0 then the lines are coplanar.

Problems:1. Find the Length of SD between the lines

$$\frac{x-1}{2} = \frac{y+2}{-1} = \frac{z-3}{-2} \text{ and } \frac{x-2}{1} = \frac{y+1}{-2} = \frac{z-1}{1}$$

Solution: Given lines are

$$\frac{x-1}{2} = \frac{y+2}{-1} = \frac{z-3}{-2} \text{ and } \frac{x-2}{1} = \frac{y+1}{-2} = \frac{z-1}{1}$$

Let (x_1, y_1, z_1) = (1,-2,3) B = (x_2, y_2, z_2) = (2,-1,1)

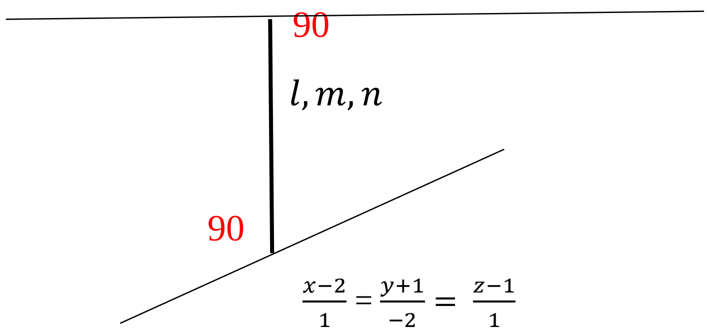
and the d.rs of the lines

$$l_1, m_1, n_1 = 2, -1, -2 \quad and \quad l_2, m_2, n_2 = 1, -2, 1$$

Suppose l, m, n *are direction cosines of the SD line* .

Since the SD line is perpendicular to given lines

$$\frac{x-1}{2} = \frac{y+2}{-1} = \frac{z-3}{-2}$$



$\therefore$ *Condition for perpendicular lines* $l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$

$\therefore \quad l(2) + m(-1) + n(-2) = 0$ -----------(1)

$l(1) + m(-2) + n(1) = 0$ -------------(2)

To solve (1) & (2)

$$\begin{matrix} -1 & -2 & 2 & -1 \\ -2 & 1 & 1 & -2 \end{matrix}$$

$$\frac{l}{-1-4} = \frac{m}{-2-2} = \frac{n}{-4+1} \Rightarrow \frac{l}{-5} = \frac{m}{-4} = \frac{n}{-3} \Rightarrow \frac{l}{5} = \frac{m}{4} = \frac{n}{3}$$

Also $\sqrt{5^2 + 4^2 + 3^2} = \sqrt{25 + 16 + 9} = \sqrt{50} = 5\sqrt{2}$

$$l = \frac{5}{5\sqrt{2}}, m = \frac{4}{5\sqrt{2}}, \quad n = \frac{3}{5\sqrt{2}}$$

Length of SD = $l(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)$

$$= \frac{5}{5\sqrt{2}}(2-1) + \frac{4}{5\sqrt{2}}(-1+2) + \frac{3}{5\sqrt{2}}(1-3)$$

$$= \frac{5}{5\sqrt{2}}(1) + \frac{4}{5\sqrt{2}}(1) + \frac{3}{5\sqrt{2}}(-2) = \frac{3}{5\sqrt{2}}$$

Problems:2. Find the Length of SD between the lines

$$\frac{x-2}{3} = \frac{y-3}{4} = \frac{z-1}{2} \quad and \quad \frac{x-4}{4} = \frac{y-5}{5} = \frac{z-2}{3}$$

Solution: Given lines are

$$\frac{x-2}{3} = \frac{y-3}{4} = \frac{z-1}{2} \quad and \quad \frac{x-4}{4} = \frac{y-5}{5} = \frac{z-2}{3}$$

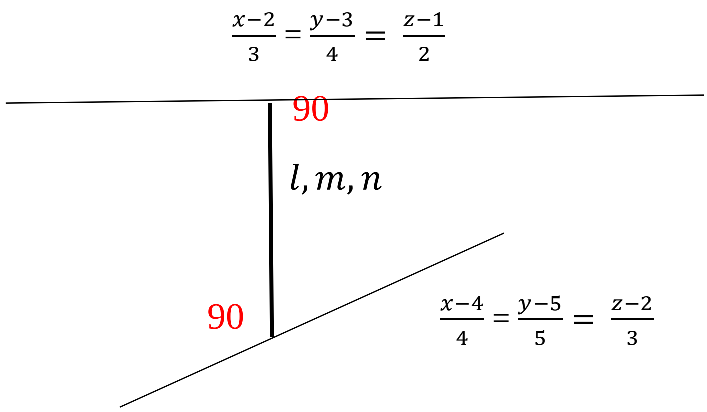
Let $(x_1, y_1, z_1) = (2, 3, 1)$ B = $(x_2, y_2, z_2) = (4, 5, 2)$

and the d.rs of the lines

$$l_1, m_1, n_1 = 3, 4, 2 \quad and \quad l_2, m_2, n_2 = 4, 5, 3$$

Suppose l, m, n *are direction cosines of the SD line* .

Since the SD line is perpendicular to given lines



$\therefore$ *Condition for perpendicular lines* $l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$

$\therefore \quad l(3) + m(4) + n(2) = 0$ -----------(1)

$l(4) + m(5) + n(3) = 0$ -------------(2)

To solve (1) & (2)

4	2	3	4
5	3	4	5

$$\frac{l}{12-10} = \frac{m}{8-9} = \frac{n}{15-16} \Rightarrow \frac{l}{2} = \frac{m}{-1} = \frac{n}{-1}$$

Also $\sqrt{2^2 + (-1)^2 + (-1)^2} = \sqrt{4+1+1} = \sqrt{6}$

$$l = \frac{2}{\sqrt{6}}, m = \frac{-1}{\sqrt{6}}, \quad n = \frac{-1}{\sqrt{6}}$$

Length of SD = $l(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)$

$$= \frac{2}{\sqrt{6}}(4-3) + \frac{-1}{\sqrt{6}}(5-4) + \frac{-1}{\sqrt{6}}(2-1)$$

$$= \frac{2}{\sqrt{6}}(1) + \frac{-1}{\sqrt{6}}(1) + \frac{-1}{\sqrt{6}}(1) = 0$$

Problems:3. Find the Length of SD between the lines

$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ *and* $\frac{x-2}{3} = \frac{y-4}{4} = \frac{z-5}{5}$ also find the equation of SD

Solution: Given lines are

$$\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4} \quad and \quad \frac{x-2}{3}=\frac{y-4}{4}=\frac{z-5}{5}$$

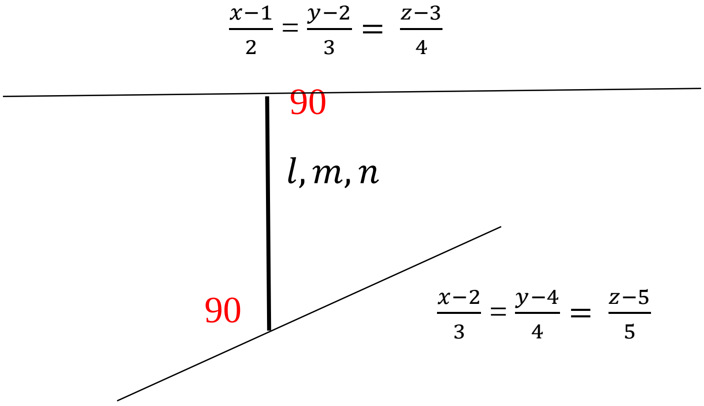
Let $(x_1, y_1, z_1) = (1, 2, 3)$ B = $(x_2, y_2, z_2) = (2, 4, 5)$

and the d.rs of the lines

$$l_1, m_1, n_1 = 2, 3, 4 \quad and \quad l_2, m_2, n_2 = 3, 4, 5$$

Suppose l, m, n *are direction cosines of the SD line* .

Since the SD line is perpendicular to given lines



$\therefore$ *Condition for perpendicular lines* $l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$

$\therefore$ $l(2) + m(3) + n(4) = 0$ -----------(1)

$l(3) + m(4) + n(5) = 0$ -------------(2)

To solve (1) & (2)

3	4	2	3
4	5	3	4

$$\frac{l}{15-16}=\frac{m}{12-10}=\frac{n}{8-9} \Rightarrow \frac{l}{-1}=\frac{m}{2}=\frac{n}{-1}$$

Also $\sqrt{(-1)^2 + 2^2 + (-1)^2} = \sqrt{1+4+1} = \sqrt{6}$

$$l = \frac{-1}{\sqrt{6}}, m = \frac{2}{\sqrt{6}}, \quad n = \frac{-1}{\sqrt{6}}$$

Length of SD = $l(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)$

$$= \frac{-1}{\sqrt{6}}(2-1) + \frac{2}{\sqrt{6}}(4-2) + \frac{-1}{\sqrt{6}}(5-3)$$

$$= \frac{-1}{\sqrt{6}}(1) + \frac{2}{\sqrt{6}}(2) + \frac{-1}{\sqrt{6}}(2) = \frac{1}{\sqrt{6}}$$

Equation of SD line is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l & m & n \end{vmatrix} = 0 = \begin{vmatrix} x-x_2 & y-y_2 & z-z_2 \\ l_2 & m_2 & n_2 \\ l & m & n \end{vmatrix}$$

$$\Rightarrow \begin{vmatrix} x-1 & y-2 & z-3 \\ 2 & 3 & 4 \\ -1 & 2 & -1 \end{vmatrix} = 0 = \begin{vmatrix} x-2 & y-4 & z-5 \\ 3 & 4 & 5 \\ -1 & 2 & -1 \end{vmatrix}$$

$\Rightarrow (x-1)(-3-8)-(y-2)(-2+4)+(z-3)(4+3)=0$

$= (x-2)(-4-10)-(y-4)(-3+5)+(z-5)(6+4)$

$\Rightarrow (x-1)(-11)-(y-2)(2)+(z-3)(7)=0$

$= (x-2)(-14)-(y-4)(2)+(z-5)(10)$

$\Rightarrow 11x+2y-7z+6=0=7x+y-5z+7$

Problems:4. Find the Length of SD between the lines

$\frac{x}{2}=\frac{y}{-3}=\frac{z}{1}$ and $\frac{x-2}{3}=\frac{y-1}{-5}=\frac{z+2}{2}$ also find the equation of SD

Solution: Given lines are

$$\frac{x}{2}=\frac{y}{-3}=\frac{z}{1} \quad and \quad \frac{x-2}{3}=\frac{y-1}{-5}=\frac{z+2}{2}$$

Let (x_1, y_1, z_1) = (0, 0, 0)   B = (x_2, y_2, z_2) = (2, 1, -2)

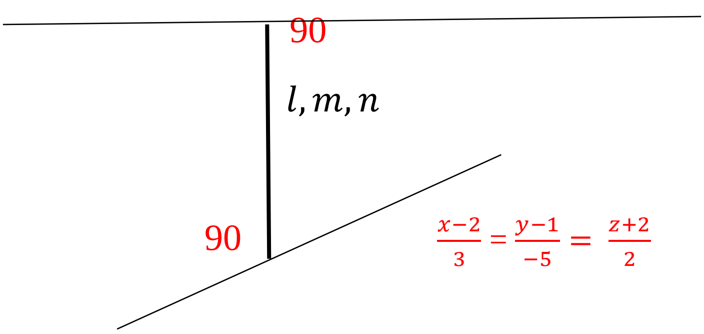
and the d.rs of the lines

$$l_1, m_1, n_1 = 2, -3, 1 \quad and \quad l_2, m_2, n_2 = 3, -5\ 2$$

Suppose l, m, n *are direction cosines of the SD line* .

Since the SD line is perpendicular to given lines

$$\frac{x}{2}=\frac{y}{-3}=\frac{z}{1}$$



$\therefore$ *Condition for perpendicular lines* $l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$

$\therefore \quad l(2) + m(-3) + n(1) = 0$ -----------(1)

$l(3) + m(-5) + n(2) = 0$ -------------(2)

To solve (1) & (2)

$$\begin{matrix} -3 & 1 & 2 & -3 \\ -5 & 2 & 3 & -5 \end{matrix}$$

$\frac{l}{-6+5} = \frac{m}{3-4} = \frac{n}{-10+9} \Rightarrow \frac{l}{-1} = \frac{m}{-1} = \frac{n}{-1} \Rightarrow \frac{l}{1} = \frac{m}{1} = \frac{n}{1}$

Also $\sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$

$$l = \frac{1}{\sqrt{3}}, m = \frac{1}{\sqrt{3}}, \ n = \frac{1}{\sqrt{3}}$$

Length of SD = $l(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)$

$$= \frac{1}{\sqrt{3}}(2-0) + \frac{1}{\sqrt{3}}(1-0) + \frac{1}{\sqrt{3}}(-2-0)$$

$$= \frac{1}{\sqrt{3}}(2) + \frac{1}{\sqrt{3}}(1) + \frac{1}{\sqrt{3}}(-2) = \frac{1}{\sqrt{3}}$$

Equation of SD line is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ l_1 & m_1 & n_1 \\ l & m & n \end{vmatrix} = 0 = \begin{vmatrix} x - x_2 & y - y_2 & z - z_2 \\ l_2 & m_2 & n_2 \\ l & m & n \end{vmatrix}$$

$$\Rightarrow \begin{vmatrix} x & y & z \\ 2 & -3 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 0 = \begin{vmatrix} x-2 & y-1 & z+2 \\ 3 & -5 & 2 \\ 1 & 1 & 1 \end{vmatrix}$$

$\Rightarrow x(-3-1) - y(2-1) + z(2+3) = 0$

$$= (x-2)(-5-2) - (y-1)(3-2) + (z+2)(3+5)$$

$\Rightarrow x(-4) - y(1) + z(5) = 0$

$$= (x-2)(-7) - (y-1)(1) + (z+2)(8)$$

$\Rightarrow 4x + y - 5z = 0 = 7x + y - 8z - 31$

Problem:5 Find the Length of SD between the lines

$\frac{x-3}{3}=\frac{y-8}{-1}=\frac{z-3}{1}$ and $\frac{x+3}{-3}=\frac{y+7}{2}=\frac{z-6}{4}$ also find the equation of SDand the points in which the SD line meets the given lines.

Solution: Let $\frac{x-3}{3}=\frac{y-8}{-1}=\frac{z-3}{1}=r \Rightarrow x=3r+3,\ y=-r+8,\ z=r+3$

Put P = $(3r+3,\ -r+8,\ r+3)$

$\frac{x+3}{-3}=\frac{y+7}{2}=\frac{z-6}{4}=r' \Rightarrow x=-3r'-3, y=2r'-7, z=4r'+6$

Put Q = $(-3r'-3, 2r'-7, 4r'+6)$

Drs of the SD line PQ

= $3r+3+3r'+3,\quad -r+8-2r'+7,\quad r+3-4r'-6$

$= 3r+3r'+6, -r-2r'+15, r-4r'-3$

But SD line PQ is perpendicular to given lines

$\frac{x-3}{3}=\frac{y-8}{-1}=\frac{z-3}{1}$ and $\frac{x+3}{-3}=\frac{y+7}{2}=\frac{z-6}{4}$

$\therefore$ $Condition\ for\ perpendicular\ lines\ l_1\, l_2+m_1m_2+n_1n_2=0$

$3(3r+3r'+6)-1(-r-2r'+15)+1(r-4r'-3)=0$

$11r+7r'=0$ -----------(1)

And -3$(3r+3r'+6)+2(-r-2r'+15)+4(r-4r'-3)=0$

$7r+29r'=0$ -----------(2)

Clearly to solve (1)&(2) $r=0\ and\ r'=0$

$\therefore$ P = (3, 8, 3) and Q = (-3, -7, 6) are the point of intersections of the given lines.

Length of SD =$\sqrt{(-3-3)^2+(-7-8)^2+(6-3)^2}$

$=\sqrt{36+225+9}=\sqrt{270}$

Drs of PQ = $3r+3r'+6, -r-2r'+15, r-4r'-3$ = 6, 15, -3

Equation of SD line $\frac{x-3}{6}=\frac{y-8}{15}=\frac{z-3}{-3} \Rightarrow \frac{x-3}{2}=\frac{y-8}{5}=\frac{z-3}{-1}$

Problem:6 Find the Length of SD between the lines

$\frac{x-3}{1}=\frac{y-5}{-2}=\frac{z-7}{1}$ and $\frac{x+1}{7}=\frac{y+1}{-6}=\frac{z+1}{1}$ also find the equation of SDand the points in which the SD line meets the given lines.

Solution: Let $\frac{x-3}{1}=\frac{y-5}{-2}=\frac{z-7}{1}=r$

Put P = $(r+3,\ -2r+5,\ r+7)$

$and\ \frac{x+1}{7}=\frac{y+1}{-6}=\frac{z+1}{1}=r' \Rightarrow x=7r'-1, y=-6r'-1, z=r'-1$

Put Q = $(-7r'-1, -6r'-5, r'-1)$

Drs of the SD line PQ

= $r+3+7r'+1,\ \ -2r+5+6r'+1,\ \ r+7-r'+1$

$= r+7r'+4, -2r+6r'+6, r-r'+8$

But SD line PQ is perpendicular to given lines

$$\frac{x-3}{1}=\frac{y-5}{-2}=\frac{z-7}{1}\ \ and\ \frac{x+1}{7}=\frac{y+1}{-6}=\frac{z+1}{1}$$

$\therefore$ *Condition for perpendicular lines* $l_1 l_2+m_1m_2+n_1n_2=0$

$1(r+7r'+4)-2(-2r+6r'+6)+1(r-r'+8)=0$

$6r-6r'=0 \Rightarrow r-r'=0$ -----------(1)

And $7(r+7r'+4)-6(-2r+6r'+6)+1(r-r'+8)=0$

$20r+12r'=0 \Rightarrow 5r+3r'=0$ -----------(2)

Clearly to solve (1) & (2) $r=0$ *and* $r'=0$

$\therefore$ P = (3, 5, 7) and Q = (-1, -1, -1) are the point of intersections of the given lines.

Length of SD =$\sqrt{(3+1)^2+(5+1)^2+(7+1)^2}$

$=\sqrt{16+36+64}=\sqrt{116}=\sqrt{4\times 29}=2\sqrt{29}$

Drs of PQ == $r+7r'+4, -2r+6r'+6, r-r'+8 = 4, 6, 8$

Equation of SD line

$$\frac{x-3}{4}=\frac{y-5}{6}=\frac{z-7}{8} \Rightarrow \frac{x-3}{2}=\frac{y-5}{3}=\frac{z-7}{4}$$

Problem:7. Find the SD and the equation of SD line between the lines

$$\frac{x}{4}=\frac{y+1}{3}=\frac{z-2}{2};\ 5x-2y-3z+6=0=x-3y+2z-3.$$

Solution: First to find the symmetric form of the line

$5x-2y-3z+6=0=x-3y+2z-3$ ----------(1)

Let l, m, n *are drs of the line*(1)

$\therefore 5l - 2m - 3n = 0$ ------(2) &

$l - 3m + 2n = 0$ --------(3)

To solve (2)&(3)   -2   -3   5   -2

-3   2   1   -3

$$\frac{l}{-4-9} = \frac{m}{-3-10} = \frac{n}{-15+2} \Rightarrow \frac{l}{-13} = \frac{m}{-13} = \frac{n}{-13} \Rightarrow \frac{l}{1} = \frac{m}{1} = \frac{n}{1}$$

Put z = 0 in (1) we get $5x - 2y + 6 = 0$

$x - 3y - 3 = 0$

-2   6   5   -2

-3   -3   1   -3

$$\frac{x}{6+18} = \frac{y}{6+15} = \frac{1}{-15+2} \Rightarrow \frac{x}{24} = \frac{y}{21} = \frac{1}{-13} \Rightarrow x = \frac{24}{-13}\ ,\ y = \frac{21}{-13}\quad z = 0$$

Symmetric form of the line $\frac{x+24/13}{1} = \frac{y+21/13}{1} = \frac{z-0}{1}$

Finally, to find length of SD between

$$\frac{x}{4} = \frac{y+1}{3} = \frac{z-2}{2};\quad \frac{x+24/13}{1} = \frac{y+21/13}{1} = \frac{z-0}{1}$$

Let (x_1, y_1, z_1) = (0, -1, 2)   B = (x_2, y_2, z_2) = (-24/13, -21/13, 0)

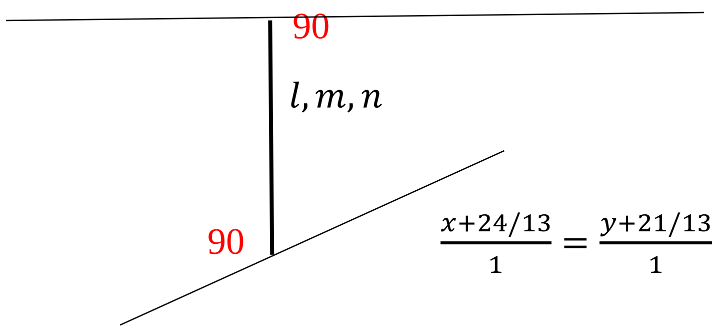
and the d.rs of the lines

$$l_1, m_1, n_1 = 4, 3, 2 \ \ and \qquad l_2, m_2, n_2 = 1, 1, 1$$

Suppose l, m, n *are direction cosines of the SD line* .

Since the SD line is perpendicular to given lines

$$\frac{x}{4} = \frac{y+1}{3} = \frac{z-2}{2}$$



$$\frac{x+24/13}{1} = \frac{y+21/13}{1} = \frac{z-0}{1}$$

$\therefore$ *Condition for perpendicular lines* $l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$

$\therefore \quad l(4) + m(3) + n(2) = 0$ -----------(1)

$l(1)+m(1)+n(1)=0$ -------------(2)

To solve (1) & (2)

3	2	4	3
1	1	1	1

$\frac{l}{3-2}=\frac{m}{2-4}=\frac{n}{4-3} \Rightarrow \frac{l}{1}=\frac{m}{-2}=\frac{n}{1}$

Also $\sqrt{(1)^2+(-2)^2+(1)^2} = \sqrt{1+4+1} = \sqrt{6}$

$$l=\frac{1}{\sqrt{6}}, m=\frac{-2}{\sqrt{6}}, \quad n=\frac{1}{\sqrt{6}}$$

Length of SD = $l(x_2-x_1)+m(y_2-y_1)+ n(z_2-z_1)$

$$=\frac{1}{\sqrt{6}}(24/13-0)+\frac{-2}{\sqrt{6}}(-21/13-1)+\frac{1}{\sqrt{6}}(0-2)$$

$$=\frac{1}{\sqrt{6}}\left(\frac{24}{13}\right)-\frac{2}{\sqrt{6}}\left(\frac{-34}{13}\right)-\frac{1}{\sqrt{6}}(2)=\frac{66}{13\sqrt{6}}$$

Equation of SD line is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l & m & n \end{vmatrix} = 0 = \begin{vmatrix} x-x_2 & y-y_2 & z-z_2 \\ l_2 & m_2 & n_2 \\ l & m & n \end{vmatrix}$$

$$\Rightarrow \begin{vmatrix} x & y+1 & z-2 \\ 4 & 3 & 2 \\ 1 & 1 & 1 \end{vmatrix} = 0 = \begin{vmatrix} x+24/13 & y+21/13 & z \\ 1 & -2 & 1 \\ 1 & 1 & 1 \end{vmatrix}$$

$\Rightarrow x(3-2)-(y+1)(4-2)+(z-2)(4-3)=0$

$$=\left(x+\frac{24}{13}\right)(-2-1)-\left(y+\frac{21}{13}\right)(1-1)+(z)(1+2)$$

$\Rightarrow x-2y+z-4=0=13x+39z-72$

Problem:8. Find the SD and the equation of SD line between the lines

$$\frac{x}{1}=\frac{y}{2}=\frac{z}{1}; x+y+2z-3=0=2x+3y+3z-4$$

Solution: First to find the symmetric form of the line

$$x+y+2z-3=0=2x+3y+3z-4 \text{ ----------(1)}$$

Let l, m, n *are drs of the line*(1)

$\therefore l + m + 2n = 0$ ------(2) &

$2l + 3m + 3n = 0$ --------(3)

To solve (2)&(3) 1 2 1 1

3 3 2 3

$$\frac{l}{3-6} = \frac{m}{4-3} = \frac{n}{3-2} \Rightarrow \frac{l}{-3} = \frac{m}{1} = \frac{n}{1}$$

Put z = 0 in (1) we get $x + y - 3 = 0$

$2x + 3y - 4 = 0$

1 -3 1 1

3 -4 2 3

$$\frac{x}{-4+9} = \frac{y}{-6+4} = \frac{1}{3-2} \Rightarrow \frac{x}{5} = \frac{y}{-2} = \frac{1}{1} \Rightarrow x = 5,\ y = -2\ \ z = 0$$

Symmetric form of the line $\frac{x-5}{-3} = \frac{y+2}{1} = \frac{z-0}{1}$

Finally, to find length of SD between

$$\frac{x}{1} = \frac{y}{2} = \frac{z}{1};\ \frac{x-5}{-3} = \frac{y+2}{1} = \frac{z-0}{1}$$

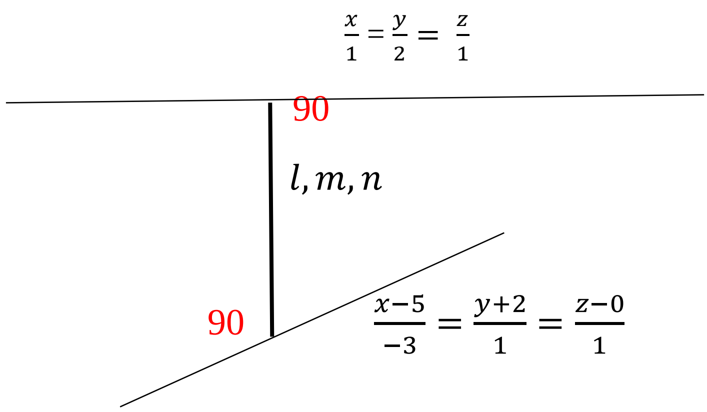
Let (x_1, y_1, z_1) = (0, 0, 0) B = (x_2, y_2, z_2) = (5, -2, 0)

and the d.rs of the lines

$$l_1, m_1, n_1 = 1, 2, 1\ \ and \qquad l_2, m_2, n_2 = -3, 1, 1$$

Suppose l, m, n *are direction cosines of the SD line* .

Since the SD line is perpendicular to given lines



$\therefore$ *Condition for perpendicular lines* $l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$

$\therefore \quad l(1) + m(2) + n(1) = 0$ -----------(1)

$$l(-3) + m(1) + n(1) = 0 \text{ -------------(2)}$$

To solve (1) & (2)

$$\begin{matrix} 2 & 1 & 1 & 2 \\ 1 & 1 & -3 & 1 \end{matrix}$$

$$\frac{l}{2-1} = \frac{m}{-3-1} = \frac{n}{1+6} \Rightarrow \frac{l}{1} = \frac{m}{-4} = \frac{n}{7}$$

Also $\sqrt{(1)^2 + (-4)^2 + (7)^2} = \sqrt{1 + 16 + 49} = \sqrt{66}$

$$l = \frac{1}{\sqrt{66}}, m = \frac{-4}{\sqrt{66}}, \ n = \frac{7}{\sqrt{66}}$$

Length of SD = $l(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)$

$$= \frac{1}{\sqrt{66}}(5 - 0) + \frac{-4}{\sqrt{66}}(-2 - 0) + \frac{7}{\sqrt{66}}(0 - 0)$$

$$= \frac{1}{\sqrt{66}}(5) - \frac{4}{\sqrt{66}}(-2) - \frac{1}{\sqrt{66}}(0) = \frac{13}{\sqrt{66}}$$

Equation of SD line is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ l_1 & m_1 & n_1 \\ l & m & n \end{vmatrix} = 0 = \begin{vmatrix} x - x_2 & y - y_2 & z - z_2 \\ l_2 & m_2 & n_2 \\ l & m & n \end{vmatrix}$$

$$\Rightarrow \begin{vmatrix} x & y & z \\ 1 & 2 & 1 \\ 1 & -4 & 7 \end{vmatrix} = 0 = \begin{vmatrix} x - 5 & y + 2 & z \\ -3 & 1 & 1 \\ 1 & -4 & 7 \end{vmatrix}$$

$$\Rightarrow x(14 + 4) - y(7 - 1) + z(-4 - 2) = 0$$

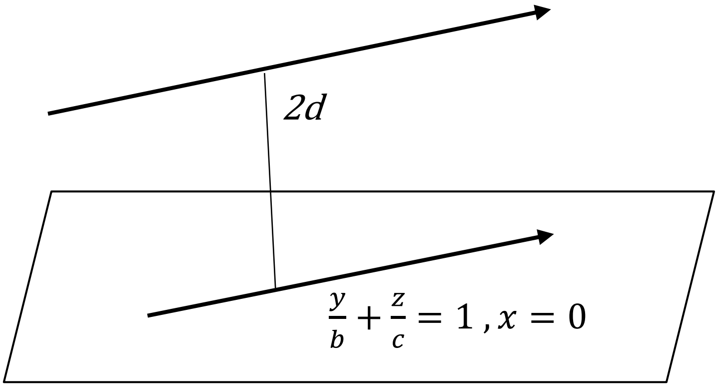
$$= (x - 5)(7 + 4) - (y + 2)(-21 - 1) + (z)(12 - 1)$$

$$\Rightarrow 18x - 6y - 6z = 0 = 13x + 22y + 11z - 21$$

Problem 9: Show that the equation to the plane containing the line $\frac{y}{b} + \frac{z}{c} = 1, x = 0$ and parallel to the line $\frac{x}{a} - \frac{z}{c} = 1, y = 0$ is $\frac{x}{a} - \frac{y}{b} - \frac{z}{c} + 1 = 0$ and if 2d is the SD Then prove that

$$\frac{1}{d^2} = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}$$

Solution: $\frac{x}{a} - \frac{z}{c} = 1\,, y = 0$



Consider *Equation of the plane containing the line*

$$\frac{y}{b} + \frac{z}{c} = 1\,, x = 0 \; is \;\; \frac{y}{b} + \frac{z}{c} - 1 + \lambda x = 0$$

$$\lambda x + \frac{y}{b} + \frac{z}{c} - 1 = 0 \text{ ----------(1)}$$

And the parallel line $\frac{x}{a} - \frac{z}{c} = 1\,, y = 0 \Rightarrow \frac{x-a}{a} = \frac{z}{c}\,, y = o$

$$\Rightarrow \frac{x-a}{a} = \frac{y-0}{0} = \frac{z}{c} \text{ ---------(2)}$$

Drs of the line = a, 0, c

Since (2) is parallel to the plane (1) condition $al + bm + cn = 0$

$$\therefore \; a(\lambda) + 0\left(\frac{1}{b}\right) + c\left(\frac{1}{c}\right) = 0 \Rightarrow a\lambda + 1 = 0 \Rightarrow \lambda = \frac{-1}{a}$$

Put the value in (1)

$$\left(\frac{-1}{a}\right)x + \frac{y}{b} - \frac{z}{c} - 1 = 0 \;\Rightarrow\; \frac{x}{a} - \frac{y}{b} - \frac{z}{c} + 1 = 0 \text{ --------(3)}$$

Given that 2d = The SD Between the given lines

= the perpendicular distance from the point (a, 0, 0) and is on the

line $\frac{x-a}{a} = \frac{y-0}{0} = \frac{z}{c}$ *to the plane* (3)

$$2d = \frac{\left|\frac{a}{a} - \frac{0}{b} - \frac{0}{c} + 1\right|}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}} = \frac{1}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}}$$

Squaring on both sides and apply cross multiplication

$$\frac{1}{d^2} = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}$$

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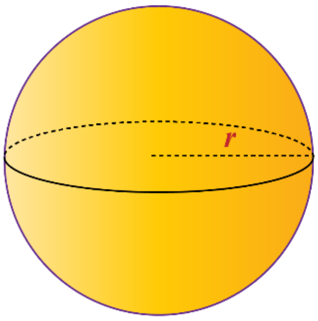
The Sphere

B. Srinivasa Rao. Lecturer in Mathematics. GDC RVPM

Definition:

The set of points in 3D-space which are at a constant distance ‘r’ from a fixed-point C is called a sphere.The fixed point is called Centre and the constant distance is called Radius of the sphere S.



Note:1. If the radius of the sphere is one unit, then it is called unit sphere.

Note:2. If the radius of the sphere is zero then it is called point sphere.

Formulas:

1.Equation of a sphere whose radius is r and centre at A= (a, b, c) is

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$$

2. Equation of a sphere whose radius is r and centre at origin is

$$x^2 + y^2 + z^2 = r^2$$

3.The general equation of a sphere is of the form

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0.$$

Centre of the sphere C = $(-u, -v, -w)$ and the radius $r = \sqrt{u^2 + v^2 + w^2 - d}$

4.If a point P = (x_1, y_1, z_1) lies on the sphere

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \text{ then}$$

$$x_1^2 + y_1^2 + z_1^2 + 2ux_1 + 2vz_1 + 2wz_1 + d = 0.$$

5.If the two or more spheres having the same centre, then the spheres are concentric spheres.

Problems:

1.Find the equation of a sphere whose centre at (2, -1,3) and the radius 5 units.

Solution: Equation of a sphere whose radius is r and centre at A= (a, b, c) is

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$$

In this problem A = (2, -1, 3) and r = 5 units

$$\therefore \quad (x-2)^2 + (y+1)^2 + (z-3)^2 = 5^2$$

$$(x^2 - 4x + 4) + (y^2 + 2y + 1) + (z^2 - 6y + 9) = 25$$

$$\Rightarrow x^2 + y^2 + z^2 - 4x + 2y - 6z - 11 = 0.$$

2.Find the equation of a sphere whose centre at (2, -3,4) and which passes through the point (1, 2, -1)

Solution: Equation of a sphere whose radius is r and centre at A= (a, b, c) is

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$$

In this problem centre A = (2, -3, 4) radius is r

$$\therefore \quad (x-2)^2 + (y+3)^2 + (z-4)^2 = r^2 --- (1)$$

But it passes through (1, 2, -1)

$$\therefore \quad (1-2)^2 + (2+3)^2 + (-1-4)^2 = r^2$$

$$\Rightarrow 1+25+25 = r^2 \Rightarrow r^2 = 51$$

$$(1) \Rightarrow \quad (x^2 - 4x + 4) + (y^2 + 6y + 9) + (z^2 - 8y + 16) = 51$$

$$\Rightarrow x^2 + y^2 + z^2 - 4x + 6y - 8z - 22 = 0.$$

3.Find the centre and radius of the sphere $x^2 + y^2 + z^2 - 6x + 2y - 4z + 14 = 0$.

Solution: In the given equation of the sphere $x^2 + y^2 + z^2 - 6x + 2y - 4z + 14 = 0$.

$$2u = -6 \Rightarrow u = -3. \;\; 2v = 2 \Rightarrow v = 1, 2w = -4 \Rightarrow w = -2 \;\&\; d = 14$$

$\therefore$ Centre = C = $(-u, -v, -w) = (3, -1, 2)$

Radius $r = \sqrt{u^2 + v^2 + w^2 - d} = \sqrt{(-3)^2 + (1)^2 + (-2)^2 - 14} = \sqrt{9 + 1 + 4 - 14} = 0$

4.Find the centre and radius of the sphere $2x^2 + 2y^2 + 2z^2 - 2x + 4y + 2z + 1 = 0$.

Solution: In the given equation of the sphere $2x^2 + 2y^2 + 2z^2 - 2x + 4y + 2z + 1 = 0$.

$\Rightarrow x^2 + y^2 + z^2 - x + 2y + z + 1/2 = 0.$

$2u = -1 \Rightarrow u = -1/2.\ \ 2v = 2 \Rightarrow v = 1, 2w = 1 \Rightarrow w = 1/2\ \&\ d = 1/2$

$\therefore$ Centre = C = $(-u, -v, -w) = (1/2, -1, -1/2)$

Radius $r = \sqrt{u^2 + v^2 + w^2 - d}$

$= \sqrt{(-1/2)^2 + (1)^2 + (1/2)^2 - 1/2} = \sqrt{1/4 + 1 + 1/4 - 1/2} = \sqrt{1}$ = 1 unit.

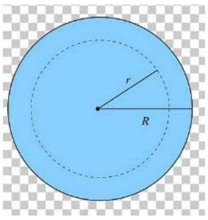
5.Find the equation of a sphere concentric with the sphere

$x^2 + y^2 + z^2 - 6x + 2y - 4z + 14 = 0$ having radius 6 units.

Solution: Centre of the sphere $x^2 + y^2 + z^2 - 6x + 2y - 4z + 14 = 0$ is (3,-1, 2)

$\therefore$ Equation of the concentric sphere centre (3, -1, 2) and radius 3 units is

$$(x - 3)^2 + (y + 1)^2 + (z - 2)^2 = 6^2$$



$$(x^2 - 6x + 9) + (y^2 + 2y + 1) + (z^2 - 4y + 4) = 36$$

$$\Rightarrow x^2 + y^2 + z^2 - 6x + 2y - 4z - 22 = 0.$$

6. Find the equation of a sphere concentric with the sphere

$x^2 + y^2 + z^2 - 2x + 4y - 6z + 7 = 0$ having radius 3 units.

Solution: Centre of the sphere $x^2 + y^2 + z^2 - 2x + 4y - 6z + 7 = 0$ is (1,-2, 3)

$\therefore$ Equation of the concentric sphere centre (1, -2, 3) and radius 3 units is

$$(x - 1)^2 + (y + 2)^2 + (z - 3)^2 = 3^2$$

$$(x^2 - 2x + 1) + (y^2 + 4y + 4) + (z^2 - 6y + 9) = 9$$

$$\Rightarrow x^2 + y^2 + z^2 - 2x + 4y - 6z + 5 = 0.$$

7.Find the equation of the sphere through the point (0, 0, 0) and making intercepts a, b, c.

Solution: Suppose the equation of the sphere

$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ ------(1)

But it passes through the points (0, 0,0) and A (a, 0, 0) B (0, b, 0) C (0, 0, c)

$\therefore$ If (0, 0,0) lies on the sphere then 0 + 0 + 0 + 0 + 0 + d = 0$\Rightarrow$ d = 0

And if (a, 0,0) lies on the sphere then

$$a^2 + 0^2 + 0^2 + 2ua + 2v(0) + 2w(0) + d = 0$$

$$\Rightarrow a^2 + 2ua + 0 = 0 \Rightarrow u = -\frac{a}{2}$$

Similarly, if we put the points B (0, b, 0) C (0, 0, c)

$$v = -\frac{b}{2}, w = -\frac{c}{2}$$

Put the values in (1) $x^2 + y^2 + z^2 + 2(-\frac{a}{2})x + 2(-\frac{b}{2})y + 2(-\frac{c}{2})z + 0 = 0$

$$\Rightarrow x^2 + y^2 + z^2 - ax - by - cz = 0 \ .$$

Note: The centre of the above sphere $(\frac{a}{2}, \frac{b}{2}, \frac{c}{2})$ and Radius $\frac{1}{2}\sqrt{a^2 + b^2 + c^2}$

8.Find the equation of the sphere through the points (4, -1,2) (0, -2,3) (1,5, -1) (2,0,1).

Solution: Suppose the equation of the required sphere

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \text{ ------(1)}$$

But it passes through the points A(4, -1,2)B (0, -2,3)C (1,5, -1)D (2,0,1)

For the first point A (4, -1,2)

$$4^2 + (-1)^2 + 2^2 + 2u(4) + 2v(-1) + 2w(2) + d = 0$$

$$16 + 1 + 4 + 8u - 2v + 4w + d = 0$$

$$8u - 2v + 4w + d = -21 \text{ ------(2)}$$

Put B (0, -2,3) in (1) $0^2 + (-2)^2 + 3^2 + 2u(0) + 2v(-2) + 2w(3) + d = 0$

$$0 + 4 + 9 + 0 - 4v + 6w + d = 0$$

$$-4v + 6w + d = -13 \text{ --------(3)}$$

Put C (1, 5, -1) in (1) $1^2 + (5)^2 + (-1)^2 + 2u(1) + 2v(5) + 2w(-1) + d = 0$

$$1 + 25 + 1 + 2u + 10v - 2w + d = 0$$

$$2u + 10v - 2w + d = -27 \text{ -------(4)}$$

Put D (2, 0, 1) in (1) $2^2 + (0)^2 + (1)^2 + 2u(2) + 2v(0) + 2w(1) + d = 0$

$$4 + 0 + 1 + 4u + 0 + 2w + d = 0$$

$$4u + 2w + d = -5 \text{ ---------(5)}$$

First to eliminate all d' in (1)(2)(3) & (4)

(2) - (3) $\Rightarrow 8u + 2v - 2w = -8 \Rightarrow 4u + v - w = -4$ -------(6)

(2)-(4)$\Rightarrow 6u - 12v + 6w = 6 \Rightarrow u - 2v + w = 1$ -----------(7)

(2)-(5)$\Rightarrow 4u - 2v + 2w = -16 \Rightarrow 2u - v + w = -8$ ----------(8)

Next to eliminate w,
$(6) + (7) \Rightarrow 5u - v = -3$ ----------(9)

$(6) + (8) \Rightarrow \ 6u = -12 \ \Rightarrow \ \boldsymbol{u = -2}$

Put the value in (9) $5(-2) - v = -3 \Rightarrow -v = -3 + 10 = 7 \ \Rightarrow \boldsymbol{v = -7}$

$$put \ u = -2 \ \& \ v = -7 \ \ in \ (7)$$

$-2 - 2(-7) + w = 1 \ \Rightarrow -2 + 14 + w = 1 \Rightarrow$ **w = -11**

$from \ (5) \ \ 4u + 2w + d = -5 \ \Rightarrow \ 4(-2) + 2(-11) + d = -5 \Rightarrow d = 25$

$\therefore$ The equation of the required sphere (1)

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$$

$$x^2 + y^2 + z^2 + 2(-2)x + 2(-7)y + 2(-11)z + 25 = 0$$

$$x^2 + y^2 + z^2 - 4x - 14y - 22z + 25 = 0$$

9.Find the equation of the sphere through the points (1, -3, 4) (1, -5, 2) (1, -3, 0) and the centre lies on the plane $\boldsymbol{x + y + z = 0}$.

Solution: Suppose the equation of the required sphere

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \text{ ------(1)}$$

But it passes through the points (1, -3, 4) (1, -5, 2) (1, -3, 0)

For the first point (1, -3, 4)

$$1^2 + (-3)^2 + 4^2 + 2u(1) + 2v(-3) + 2w(4) + d = 0$$

$$1 + 9 + 16 + 2u - 6v + 8w + d = 0$$

$$2u - 6v + 8w + d = -26 \text{ ------(2)}$$

Put (1, -5, 2) in (1) $\ 1^2 + (-5)^2 + 2^2 + 2u(1) + 2v(-5) + 2w(2) + d = 0$

$$1 + 25 + 4 + 2u - 10v + 4w + d = 0$$

$$2u - 10v + 4w + d = -30 \text{ --------(3)}$$

Put(1, -3, 0) in (1) $\ 1^2 + (-3)^2 + (0)^2 + 2u(1) + 2v(-3) + 2w(0) + d = 0$

$$1 + 9 + 0 + 2u - 6v - 0 + d = 0$$

$$2u - 6v + d = -10 \text{ -------(4)}$$

And the centre $(-u, -v, -w)$ *lies on the plane* $\ x + y + z = 0$

$\therefore\ -u-v-w=0 \Rightarrow u+v+w=0$ ---------(5)

First to eliminate all d' in (1)(2)(3) & (4)

(2) - (3) $\Rightarrow 4v+4w=4 \ \Rightarrow v+w=1$ -------(6)

(2)-(4)$\Rightarrow 8w=-16 \Rightarrow \boldsymbol{w}= \ \mathbf{-2}$

From (5) & (6) $u+1=0 \ \Rightarrow \boldsymbol{u}=\mathbf{-1}$

From (5) $u+v+w=0 \ \Rightarrow \ -1+v+(-2)=0 \ \Rightarrow \boldsymbol{v}=\mathbf{3}$

From (4) $2u-6v+d=-10 \ \Rightarrow 2(-1)-6(3)+d=-10 \Rightarrow \boldsymbol{d}=\mathbf{10}$

$\therefore$ The equation of the required sphere $x^2+y^2+z^2+2ux+2vy+2wz+d=0$

$$x^2+y^2+z^2+2(-1)x+2(3)y+2(-2)z+10=0$$

$$x^2+y^2+z^2-2x+6y-4z+10=0$$

10.Find the equation of the sphere through the points (1, 0,0) (0, 1, 0) (0, 0, 1) and having the least radius.

Solution: Suppose the equation of the required sphere

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \text{ ------(1)}$$

But it passes through the points (1, 0, 0) (0, 1, 0) (0, 0, 1)

$\therefore \quad 1^2+0^2+0^2+2u(1)+2v(0)+2w(0)+d=0$

$$1+2u+d=0 \quad \Rightarrow \quad u=-\frac{(d+1)}{2}$$

Similarly for the remaining points (0,1,0) (0,0,1)

$$v=-\frac{(d+1)}{2} \quad \& \quad w=-\frac{(d+1)}{2}$$

Given that the sphere having least radius

Radius $r=\sqrt{u^2+v^2+w^2-d}$

$$=\sqrt{\left[-\frac{(d+1)}{2}\right]^2+\left[-\frac{(d+1)}{2}\right]^2+\left[-\frac{(d+1)}{2}\right]^2-d} = \sqrt{3\left[-\frac{(d+1)}{2}\right]^2-d}$$

$$r=\sqrt{\frac{3(d+1)^2}{4}-d} \Rightarrow r^2=\frac{3(d+1)^2}{4}-d$$

$$Let\ f(d)=\frac{3(d+1)^2}{4}-d$$

$$f'(d)=\frac{6(d+1)}{4}-1 \ and\ f''(d)=\frac{6}{4}-0=\frac{3}{2}>0$$

$If\ \ f'(d) = \frac{6(d+1)}{4} - 1 = 0\ \Rightarrow 3(d+1) = 4\ \Rightarrow d = -\frac{1}{3}\ \ also\ f''\left(-\frac{1}{3}\right) = \frac{3}{2} > 0$

$\therefore$ At $d = -\frac{1}{3}$ *the sphere has least radius*

Now $u = -\ \frac{(d+1)}{2}\ = u = -\ \frac{\left(-\frac{1}{3}+1\right)}{2} = \ -\frac{(\frac{2}{3})}{2}\ \ = -\frac{1}{3} \Rightarrow\ u = -\frac{1}{3}$

Similarly, $v = -\frac{1}{3}$ & $w = -\frac{1}{3}$

Hence equation of the required sphere $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$

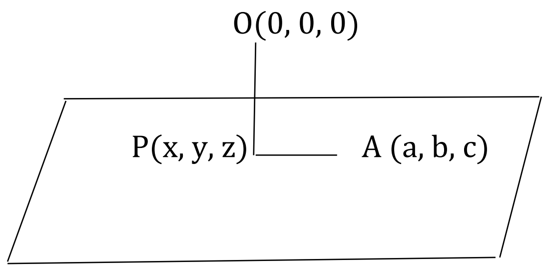
$x^2 + y^2 + z^2 + 2(-\frac{1}{3})x + 2(-\frac{1}{3})y + 2(-\frac{1}{3})z + (-\frac{1}{3}) = 0$

$$3(\ x^2 + y^2 + z^2) - 2(\ x + y + z) - 1 = 0$$

11.A plane passes through the fixed point (a, b, c). Show that the foot of the perpendicular from origin to the plane lies on the sphere

$$\boldsymbol{x^2 + y^2 + z^2 - ax - by - cz = 0.}$$

Solution:



Let P=(x, y, z) is the foot of the perpendicular from O(0,0,0) to the plane.

But the plane passes through A(a, b, c).

Direction ratios of OP = $x - 0, y - 0, z - 0 = x, y, z$

And the direction ratios of AP = $x - a, y - b, z - c$

But OP and AP are perpendicular

$\therefore\ \ x(x - a) + y(y - b) + z(z - c)$

$\Rightarrow x^2 + y^2 + z^2 - ax - by - cz = 0$ *is the locus of foot of the perpendicular*

$$P = (x, y, z)$$

12. A plane passes through the fixed point (a, b, c). and intersects the axes in A, B, C. Show that the centre of the sphere OABC lies on $\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 2.$

Solution: Let the equation of the plane $\frac{x}{\alpha} + \frac{y}{\beta} + \frac{z}{\gamma} = 1$

whose intercepts are A = $(\alpha, 0, 0)\ \ B = (0, \beta, 0)\ and\ C = (0, 0, \gamma)$

But it passes through the fixed point (a, b, c)

$$\therefore \frac{a}{\alpha} + \frac{b}{\beta} + \frac{c}{\gamma} = 1 \text{ --------(1)}$$

We know that the equation of a sphere through

O(0, 0, 0) $A = (\alpha, 0,0)$ $B = (0, \beta, 0)$ and $C = (0,0,\gamma)$ is

$$x^2 + y^2 + z^2 - \alpha x - \beta y - \gamma z = 0.$$

Centre of the sphere P $= (\frac{\alpha}{2}, \frac{\beta}{2}, \frac{\gamma}{2}) = (x_1, y_1, z_1)$

$$\Rightarrow \frac{\alpha}{2} = x_1 \Rightarrow \alpha = 2x_1 \quad \frac{\beta}{2} = y_1 \Rightarrow \beta = 2y_1 \text{ and } \frac{\gamma}{2} = z_1 \Rightarrow \gamma = 2z_1$$

Put the values in (1) $\frac{a}{2x_1} + \frac{b}{2y_1} + \frac{c}{2z_1} = 1 \Rightarrow \frac{a}{x_1} + \frac{b}{y_1} + \frac{c}{z_1} = 2$

Hence the locus of the centre $P(x_1, y_1, z_1)$ *is* $\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 2$

13.A sphere of constant radius k passes through origin and intersects the coordinate axes in A, B, C. Prove that the centroid of the Δ ABC lies on the sphere

$$9(x^2 + y^2 + z^2) = 4k^2$$

Solution: We know that the equation of a sphere through

O (0, 0, 0) $A = (\alpha, 0,0)$ $B = (0, \beta, 0)$ and $C = (0,0,\gamma)$ is

$$x^2 + y^2 + z^2 - \alpha x - \beta y - \gamma z = 0$$

Centre $= (\frac{\alpha}{2}, \frac{\beta}{2}, \frac{\gamma}{2})$

Radius r $= \frac{1}{2}\sqrt{\alpha^2 + \beta^2 + \gamma^2} = k$ (Given)

Squaring on both sides $\alpha^2 + \beta^2 + \gamma^2 = 4k^2$ ----------(1)

Now the centroid of the $\Delta ABC = (\frac{\alpha+0+0}{3}, \frac{0+\beta+0}{3}, \frac{0+0+\gamma}{3}) = (x_1, y_1, z_1)$ say

$$\alpha = 3x_1 \quad \beta = 3y_1 \quad \gamma = 3z_1$$

Put the values in (1) $(3x_1)^2 + (3y_1)^2 + (3z_1)^2 = 4k^2$

$$\Rightarrow 9{x_1}^2 + 9{y_1}^2 + 9{z_1}^2 = 4k^2$$

Locus of the centre is $9(x^2 + y^2 + z^2) = 4k^2$

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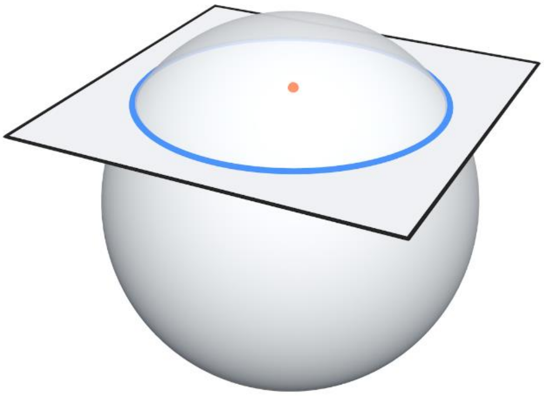
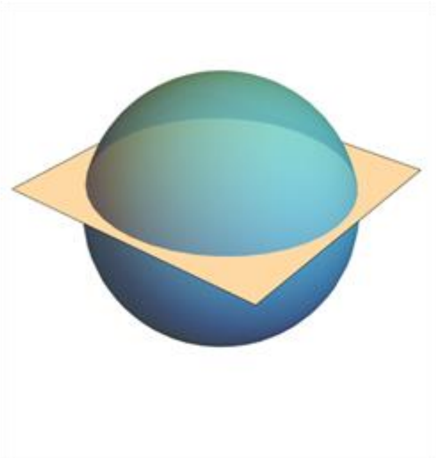
Plane section of a sphere

1.The plane section of a sphere is a circle.

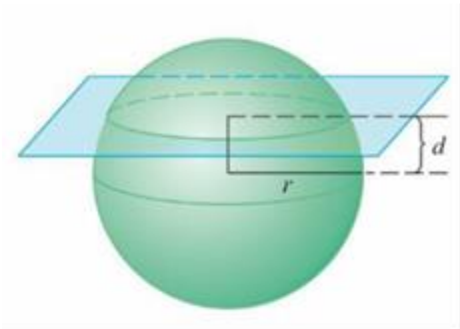


2.there are two types of circles in the plane section of sphere A) Great circle B) Small circle.

A) If the centre of the sphere lies on the section plane, then it is a great circle.



B) If the centre of the sphere does not lie on the section plane, then it is a small circle



3.Equation of a sphere through the circle S= 0 = π is S + $\lambda\pi = 0$ *where λ is a constant.*

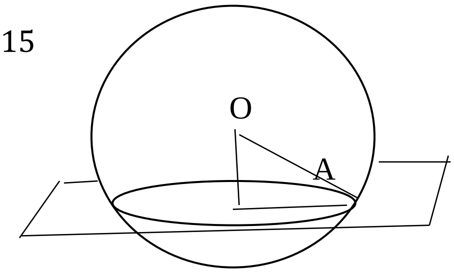
Problems:1. Find the centre and radius of the circle

$$\boldsymbol{x^2 + y^2 + z^2 - 169 = 0 = x - 2y + 2z - 15.}$$

Solution:

Consider the given circle

$$x^2 + y^2 + z^2 - 169 = 0 = x - 2y + 2z - 15$$



O = centre of the sphere = (0,0,0) OA= radius = $\sqrt{169} = 13$

OP = The perpendicular distance from O(0,0,0) to the plane

$$x - 2y + 2z - 15 = 0$$

$$= \frac{|-15|}{\sqrt{1+4+4}} = \frac{15}{3} = 5$$

Now in the right-angle triangle OPA

$PA^2 = 0A^2 - OP^2 = 169 - 25 = 144.$

Radius of the circle = PA = 12

Drs OP = Drs of the normal line to the plane $x - 2y + 2z - 15 = 0$

= 1, -2, 2

Equation of the line OP is $\frac{x-0}{1} = \frac{y-0}{-2} = \frac{z-0}{2} = r\ (say) \Rightarrow x = r, y = -2r, z = 2r$

Let P = $(r, -2r, 2r)$ *but it is on the plane* $x - 2y + 2z - 15 = 0$

$$\therefore\ r - 2(-2r) + 2(2r) - 15 = 0 \Rightarrow 9r = 15 \Rightarrow r = \frac{5}{3}.$$

Centre of the circle P = $(r, -2r, 2r) = (5/3, -2(\frac{5}{3}), 2\left(\frac{5}{3}\right)) = (\frac{5}{3}, \frac{-10}{3}, \frac{10}{3})$

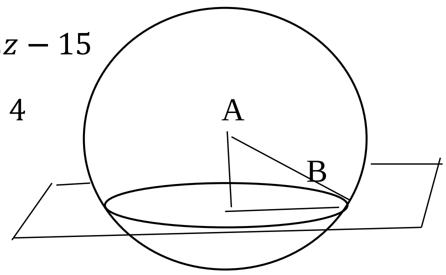
2. Problems:1. Find the centre and radius of the circle

$$\boldsymbol{x^2 + y^2 + z^2 - 2y - 4z - 11 = 0 = x + 2y + 2z - 15.}$$

Solution:

Consider the given circle

$$x^2 + y^2 + z^2 - 2y - 4z - 11 = 0 = x + 2y + 2z - 15$$



A = centre of the sphere = (0,1,2) AB= radius = $\sqrt{0 + 1 + 4 + 11} = 4$

AP = The perpendicular distance from O(0,1,2) to the plane

$$x + 2y + 2z - 15 = 0$$

$$= \frac{|0+2+4-15|}{\sqrt{1+4+4}} = \frac{9}{3} = 3$$

Now in the right-angle triangle APB

$PB^2 = AB^2 - AP^2 = 16 - 9 = 7 \Rightarrow PB = \sqrt{7}.$

$\therefore$ Radius of the circle = PB = $\sqrt{7}$

Drs OP = Drs of the normal line to the plane $x + 2y + 2z - 15 = 0$

= 1, 2, 2

Equation of the line OP is $\frac{x-0}{1} = \frac{y-1}{2} = \frac{z-2}{2} = r\ (say) \Rightarrow x = r, y = 2r + 1, z = 2r + 2$

Let P = $(r, 2r + 1, 2r + 2)$ *but it is on the plane* $x + 2y + 2z - 15 = 0$

$\therefore\ r + 2(2r + 1) + 2(2r + 2) - 15 = 0 \Rightarrow 9r = 9 \Rightarrow r = 1.$

Centre of the circle P = $(r, 2r + 1, 2r + 2) = (1, 2 + 1, 2 + 2) = (1,3,4)$

****3.Find the equation of the sphere through the circle**

$x^2 + y^2 + z^2 - 9 = 0 = 2x + 3y + 4z - 5$ and the point (1,2,3)

Solution: Let S = $x^2 + y^2 + z^2 - 9 = 0$ *and* $\pi = 2x + 3y + 4z - 5$ =0

We know that Equation of a sphere through the circle

S= $0 = \pi$ is $S + \lambda\,\pi = 0$ *where* λ *is a constant.*

$\therefore x^2 + y^2 + z^2 - 9 + \lambda(2x + 3y + 4z - 5) = 0$ --------(1)

But it is passes through the point (1,2,3)

$\therefore\ 1^2 + 2^2 + 3^2 - 9 + \lambda(2(1) + 3(2) + 4(3) - 5) = 0 \Rightarrow 5 + 15\lambda = 0 \Rightarrow \lambda = -1/3$

Put the value in (1) $\quad x^2 + y^2 + z^2 - 9 - \frac{1}{3}(2x + 3y + 4z - 5) = 0$

$3x^2 + 3y^2 + 3z^2 - (2x + 3y + 4z) - 5) - 22 = 0$ is the equation of required sphere.

4.Show that the two circles

$$x^2 + y^2 + z^2 - y + 2z = 0 = x - y + z - 2$$

$$x^2 + y^2 + z^2 + x - 3y + z - 5 = 0 = 2x - y + 4z - 1$$

lie on the same sphere and find its equation.

Solution: Let S = $x^2 + y^2 + z^2 - y + 2z = 0$ *and* $\pi = x - y + z - 2$ =0

We know that Equation of a sphere through the circle

S= 0 = π is S + $\lambda\,\pi = 0$ *where* λ *is a constant.*

$\therefore x^2 + y^2 + z^2 - y + 2z + \lambda(x - y + z - 2) = 0$

$x^2 + y^2 + z^2 + \lambda x - (1 + \lambda)y + (2 + \lambda)z - 2\lambda = 0$ ----------(1)

Let S' = $x^2 + y^2 + z^2 + x - 3y + z - 5 = 0$ *and* $\pi' = 2x - y + 4z - 1$ =0

We know that Equation of a sphere through the circle

S' = 0 = π' is S' + $\mu\,\pi' = 0$ *where* μ *is a constant.*

$\therefore x^2+y^2+z^2+x-3y+z-5+\mu(2x-y+4z-1)=0$

$x^2+y^2+z^2+(1+2\mu)x-(3+\mu)y+(1+4\mu)z-(5+\mu)=0$--------(2)

If the circles lie in the same sphere to comparing (1) & (2)

$\lambda=1+2\mu \Rightarrow \lambda-2\mu=1$ --------(3)

$-(1+\lambda)=-(3+\mu) \Rightarrow \lambda-\mu=2$ ------(4)

$2+\lambda=1+4\mu \Rightarrow \lambda-4\mu=-1$ -----(5)

(3) - (4) ⇒ $-\mu$ = -1 ⇒ μ = 1

From (4) $\lambda-\mu=2 \Rightarrow \lambda-1=2 \Rightarrow \lambda=3$

Put the values $\lambda=3$ & $\mu=1$ in Equation (5)

$LHS=\lambda-4\mu=(3)-4(1)=-1\ RHS$

∴ The given circles lie in the same sphere and its equation is (2)

$$x^2+y^2+z^2+(1+2\mu)x-(3+\mu)y+(1+4\mu)z-(5+\mu)=0$$

$$x^2+y^2+z^2+(1+2)x-(3+1)y+(1+4)z-(5+1)=0$$

$$x^2+y^2+z^2+3x-4y+5z-6=0$$

5.Show that the two circles

$$\boldsymbol{x^2+y^2+z^2-2x+3y+4z-5=0=5y+6z+1}$$

$$\boldsymbol{x^2+y^2+z^2-3x-4y+5z-6=0=x+2y-7z}$$

lie on the same sphere and find its equation.

Solution: Let S = $x^2+y^2+z^2-2x+3y+4z-5$ *and* $\pi=5y+6z+1$ =0

We know that Equation of a sphere through the circle

S= 0 = π is S + $\lambda\,\pi=0$ *where* λ *is a constant.*

$\therefore x^2+y^2+z^2-2x+3y+4z-5+\lambda(5y+6z+1)=0$

$x^2+y^2+z^2-2x+(3+5\lambda)y+(4+6\lambda)z-(5-\lambda)=0$ ----------(1)

Let S' =$x^2+y^2+z^2-3x-4y+5z-6=0$ *and* $\pi'=x+2y-7z$ =0

We know that Equation of a sphere through the circle

S' = 0 = π' is S' + $\mu\,\pi'=0$ *where* μ *is a constant.*

$\therefore x^2+y^2+z^2-3x-4y+5z-6+\mu(x+2y-7z)=0$

$x^2+y^2+z^2-(3-\mu)x-(4-2\mu)y+(5-7\mu)z-6=0$--------(2)

If the circles lie in the same sphere to comparing (1) & (2)

$-2=-(3-\mu) \Rightarrow \mu=1$ --------(3)

$$(3+5\lambda) = -(4-2\mu) \Rightarrow 5\lambda - 2\mu = -7 \text{ ------(4)}$$

$$4+6\lambda = 5-7\mu \Rightarrow 6\lambda + 7\mu = 1 \text{ -----(5)}$$

From (4) $5\lambda - 2\mu = -7 \Rightarrow 5\lambda - 2(1) = -7 \Rightarrow 5\lambda = -5 \Rightarrow \lambda = -1$

Put the values $\lambda = -1$ & $\mu = 1$ in Equation (5)

$LHS = 6\lambda + 7\mu = 6(-1) + 7(1) = 1\ RHS$

∴ The given circles lie in the same sphere and its equation is (2)

$$x^2+y^2+z^2-(3-1)x-(4-2)y+(5-7)z-6=0$$

$$x^2+y^2+z^2-2x-2y-2z-6=0$$

****6. Find the equations of spheres passing through the circle**

$\boldsymbol{x^2+y^2+z^2-4=0=z}$ ***and is intersected by the plane***

$\boldsymbol{x+2y+2z=0}$ ***in circle of radius*** $\boldsymbol{3}$.

Solution: Equation of sphere passing through the circle $x^2+y^2+z^2-4=0=z$

$$x^2+y^2+z^2-4+\lambda z = 0$$

$$x^2+y^2+z^2+\lambda z-4=0 \text{ --------(1)}$$

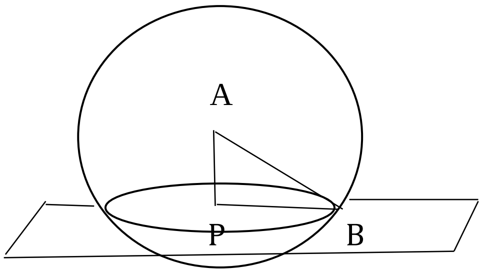
Centre A= $(0, 0, -\frac{\lambda}{2})$ and the radius AB = $\sqrt{0+0+\frac{\lambda^2}{4}+4} = \sqrt{\frac{\lambda^2}{4}+4}$

Given that $x+2y+2z=0$ *intersects the shere* (1)

PB = Radius of intersecting circle = 3

Now AP = The perpendicular distance from the centre A= $(0, 0, -\frac{\lambda}{2})$ of the sphere to the plane $x+2y+2z=0$

$$= \frac{|0+0+2\left(-\frac{\lambda}{2}\right)|}{\sqrt{1+4+4}} = \frac{\lambda}{3}.$$



In the right-angle triangle APB

$$AB^2 = AP^2 + PB^2$$

$$\Rightarrow \frac{\lambda^2}{4}+4 = \left(\frac{\lambda}{3}\right)^2 + 3^2 \text{ P}$$

$$\Rightarrow \frac{\lambda^2}{4} - \frac{\lambda^2}{9} = 9-4=5 \Rightarrow \frac{5\lambda^2}{36} = 5 \Rightarrow \lambda^2 = 36 \Rightarrow \lambda = \pm 6$$

Put the value in (1) $x^2+y^2+z^2 \pm 6z - 4 = 0$

***6. Find the equation of a sphere for which the circle $\boldsymbol{x^2+y^2+z^2+7y-2z+2=0}$**

$\boldsymbol{2x+3y+4z-8=0}$ is a great circle.

Solution: Equation of sphere passing through the circle is

$$x^2 + y^2 + z^2 + 7y - 2z + 2 + \lambda(2x + 3y + 4z - 8) = 0$$

$$\Rightarrow x^2 + y^2 + z^2 + 2\lambda x + (7 + 3\lambda)y - (2 - 4\lambda)z + (2 - 8\lambda) = 0 \text{ ----------(1)}$$

Centre of the sphere $\left(-\frac{2\lambda}{2}, -\frac{7+3\lambda}{2}, \frac{2-4\lambda}{2}\right) = \left(-\lambda, -\frac{7+3\lambda}{2}, \frac{1-2\lambda}{1}\right)$

But the given circle is great circle

$\therefore$ Centre of the sphere lies on $2x + 3y + 4z - 8 = 0$

$$\therefore 2(-\lambda) + 3(-\frac{7+3\lambda}{2}) + 4(1 - 2\lambda) - 8 = 0$$

$$-4\lambda - 21 - 9\lambda + 8 - 16\lambda - 16 = 0 \Rightarrow -29\lambda - 29 = 0 \Rightarrow \lambda = -1$$

Equation of the required sphere (1)

$$x^2 + y^2 + z^2 + 2\lambda x + (7 + 3\lambda)y - (2 - 4\lambda)z + (2 - 8\lambda) = 0$$

Put $\lambda = -1$ $\quad x^2 + y^2 + z^2 - 2x + 4y - 6z + 10 = 0$

6.Find the equation of a sphere for which the circle

$$\boldsymbol{x^2 + y^2 + z^2 - 4x + 6y - 8z + 4 = 0 = x + y + z - 3}$$

is a great circle.

Solution: Equation of sphere passing through the circle is

$$x^2 + y^2 + z^2 - 4x + 6y - 8z + 4 + \lambda(x + y + z - 3) = 0$$

$$\Rightarrow x^2 + y^2 + z^2 - (4 - \lambda)x + (6 + \lambda)y - (8 - \lambda)z + (4 - 3\lambda) = 0 \text{ ----------(1)}$$

Centre of the sphere $\left(\frac{4-\lambda}{2}, -\frac{6+\lambda}{2}, \frac{8-\lambda}{2}\right)$

But the given circle is great circle

$\therefore$ Centre of the sphere lies on $x + y + z - 3 = 0$

$$\frac{4-\lambda}{2} - \frac{6+\lambda}{2} + \frac{8-\lambda}{2} - 3 = 0 \Rightarrow \frac{4-\lambda-6-\lambda+8-\lambda-6}{2} = 0 \Rightarrow \lambda = 0$$

Equation of the required sphere (1) $x^2 + y^2 + z^2 - 4x + 6y - 8z + 4 = 0.$

****

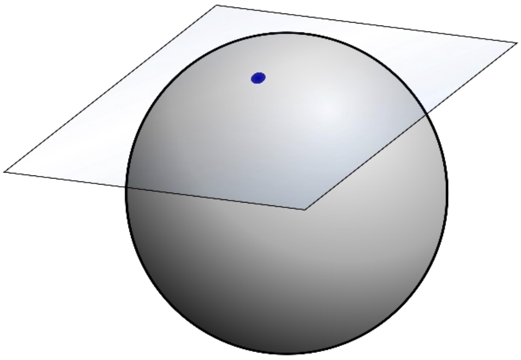
Tangent plane to the sphere

1.Condition for a plane $ax + by + cz + d = 0$ *tauches a sphere*

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \text{ is}$$

Radius = The perpendicular distance from the centre of the sphere to the plane.

That is $r = d$.



2.Equation of the tangent plane to the sphere

S = $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ at the point P=$(x_1, y_1, z_1)$is

S_1= $xx_1 + yy_1 + zz_1 + u(x + x_1) + v(y + y_1) + w(z + z_1) + d = 0$.

3.Length of tangent from (x_1, y_1, z_1)in to the sphere S = 0 is $\sqrt{S_{11}}$

4.Conditions for touching two spheres:

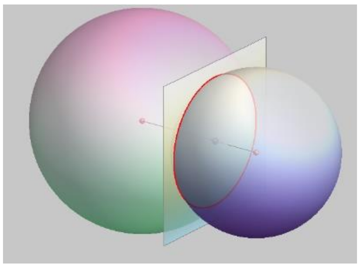
If C_1, C_2 *are centres and* r_1, r_2 *are radii of two spheres respectively*.

(i)If $C_1 C_2 = r_1 + r_2$ then the spheres are touch externally and the point of contact is

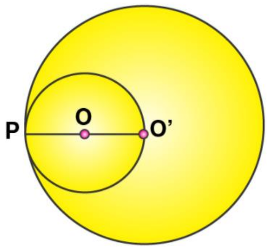
Divides C_1, C_2 in the ratio $r_1 : r_2$ internally.

(ii)If $C_1 C_2 = | r_1 - r_2 |$ then the spheres are touch internally and the point of contact is

Divides C_1, C_2 in the ratio $r_1 : r_2$ externally.



Externally



Internally

Problems:

1.Find the length of the tangent line to the sphere $x^2 + y^2 + z^2 - 3x + 5y + 7 = 0$ from the point (3, 1, -1).

Solution:

We know that the. Length of tangent from (x_1, y_1, z_1) into the sphere S = 0 is $\sqrt{S_{11}}$

Let S = $x^2 + y^2 + z^2 - 3x + 5y + 7$ and (x_1, y_1, z_1) = (3,1, -1).

Now $S_{11} = x_1^2 + y_1^2 + z_1^2 - 3x_1 + 5y_1 + 7 = 3^2 + 1^2 + (-1)^2 - 3(3) + 5(1) + 7 = 14$

$\therefore$ Length of tangent line = $\sqrt{S_{11}} = \sqrt{14}$.

2.Find the points of intersections of the line $\frac{x-8}{4} = \frac{y}{1} = \frac{z-1}{-1}$ *and the sphere*

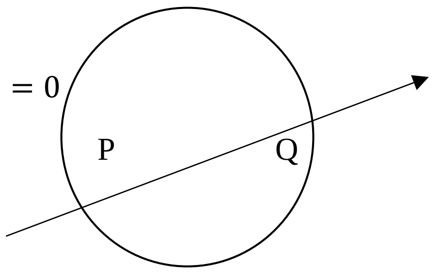
$x^2 + y^2 + z^2 - 4x + 6y - 2z + 5 = 0.$

Solution: Let $\frac{x-8}{4} = \frac{y}{1} = \frac{z-1}{-1} = r \Rightarrow x = 4r + 8, y = r, z = -r + 1$

Let P = $(4r + 8, r, -r + 1)$

$\because$ The line intersects the sphere

$\Rightarrow$ P lies on the sphere $x^2 + y^2 + z^2 - 4x + 6y - 2z + 5 = 0$



$\therefore (4r + 8)^2 + r^2 + (-r + 1)^2 - 4(4r + 8) + 6r - 2(-r + 1) + 5 = 0$

$(16r^2 + 64r + 64) + r^2 + r^2 - 2r + 1$

$-16r - 32 + 6r + 2r - 2 + 5 = 0$

$18r^2 + 54r + 36 = 0 \Rightarrow r^2 + 3r + 2 = 0 \Rightarrow r = -1, -2$

Put $r = -1 \Rightarrow$ P = $(4r + 8, r, -r + 1) = (-4 + 8, -1, 1 + 1) = (4, -1, 2)$

Put $r = -2 \Rightarrow$ P = $(4r + 8, r, -r + 1) = (-8 + 8, -2, 2 + 1) = (0, -2, 3)$

$\therefore$ The points of intersections = $(4, -1, 2)$ and $(0, -2, 3)$

3.Show that the plane $2x - 2y + z + 12 = 0$ touches the sphere

$x^2 + y^2 + z^2 - 2x - 4y + 2z - 3 = 0$ and find the point of contact.

Solution: Consider the equation of the sphere

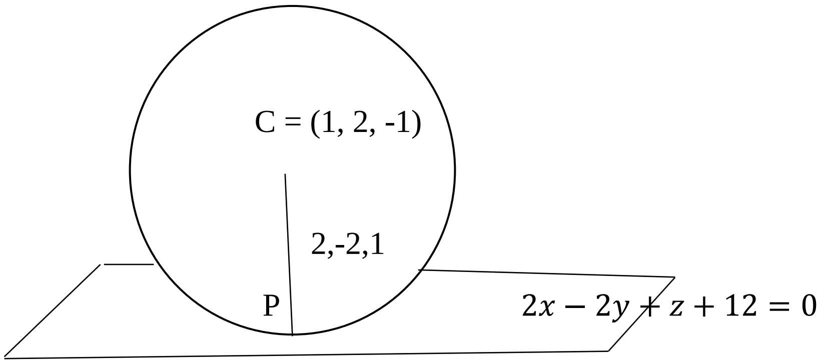
$$x^2 + y^2 + z^2 - 2x - 4y + 2z - 3 = 0$$

Centre C = (1, 2, -1) and radius r = $\sqrt{1 + 4 + 1 + 3} = \sqrt{9} = 3$

The perpendicular distance from the centre C = (1, 2, -1) to the plane $2x - 2y + z + 12 = 0$

$$= \frac{|2(1) - 2(2) + (-1) + 12|}{\sqrt{4 + 4 + 1}} = \frac{9}{3} = 3 = Radius$$

$\therefore$ The plane touches the sphere.



Let P is the point of contact. Drs of CP = 2, -2,1

Equation of the line CP where C = (1, 2, -1) is

$$\frac{x-1}{2} = \frac{y-2}{-2} = \frac{z+1}{1} = r\ say$$

$$\Rightarrow x = 2r+1,\ y = -2r+2,\ z = r-1$$

$$P = (\ 2r+1,\ -2r+2,\ r-1\ )$$

But P lies on the plane $2x - 2y + z + 12 = 0$

$$\therefore 2(2r+1) - 2(-2r+2) + (r-1) + 12 = 0$$

$$9r + 9 = 0 \Rightarrow r = -1$$

Point of contact P = $(\ 2r+1,\ -2r+2,\ r-1\ )$

$$= (\ 2(-1)+1,\ -2(-1)+2,\ (-1)-1\ )$$

$$P = (\ -1,\ 4, -2)$$

4.Show that the plane $2x - y - 2z - 4 = 0$ touches the sphere

$x^2 + y^2 + z^2 + 2x - 6y + 1 = 0$ and find the point of contact.

Solution: Consider the equation of the sphere

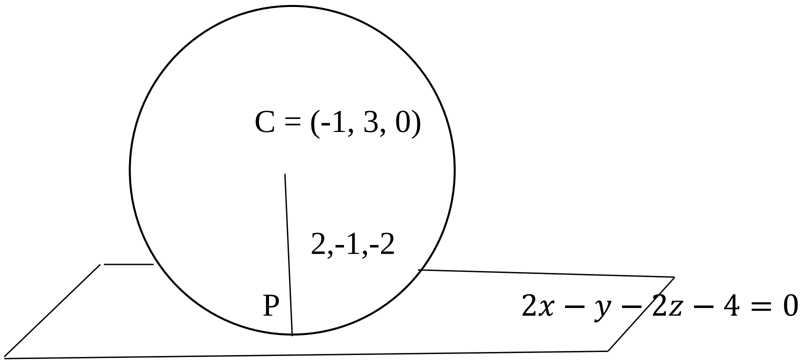
$$x^2 + y^2 + z^2 + 2x - 6y + 1 = 0$$

Centre C = (-1, 3, 0) and radius r = $\sqrt{1+9+0-1} = \sqrt{9} = 3$

The perpendicular distance from the centre C = (-1, 3, 0) to the plane $2x - y - 2z - 4 = 0$

$$= \frac{|\ 2(-1)-3-0-4\ |}{\sqrt{4+1+4}} = \frac{9}{3} = 3 = Radius$$

$\therefore$ The plane touches the sphere.



Let P is the point of contact. Drs of CP = 2, -1,2

Equation of the line CP where C = (-1, 3, 0) is

$$\frac{x+1}{2} = \frac{y-3}{-1} = \frac{z-0}{2} = r \; say$$

$$\Rightarrow x = 2r - 1, \; y = -r + 3, \; z = 2r$$

$$P = (\, 2r - 1, \; -r + 3, \; 2r \,)$$

But P lies on the plane $2x - y - 2z - 4 = 0$

$$\therefore 2(2r - 1) - (-r + 3) - 2(2r) - 4 = 0$$

$$9r - 9 = 0 \Rightarrow r = 1$$

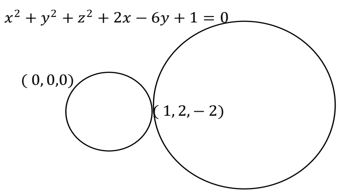
Point of contact P = $(2r - 1, \; -r + 3, \; 2r)$

$= (\; 2 - 1, \; -1 + 3, \; 2\,)$

P = $(\,1, 2, 2)$

5.Find the equation of a sphere which touches the sphere $x^2 + y^2 + z^2 + 2x - 6y + 1 = 0$ at the point $(\,1, 2, -2)$ and passes through origin.

Solution: Let S = $x^2 + y^2 + z^2 + 2x - 6y + 1 = 0$



Equation of the tangent plane to the sphere

$$S_1 = xx_1 + yy_1 + zz_1 + u(x + x_1) + v(y + y_1) + w(z + z_1) + d = 0.$$

$$P = (x_1, y_1, z_1) = (\,1, 2, -2)$$

$$S_1 = x(1) + y(2) + z(-2) + (x + 1) - 3(y + 2) + 1 = 0.$$

$$\Rightarrow 2x - y - 2z - 4 = 0$$

Equation of the sphere which touches S =0 at P is

$x^2 + y^2 + z^2 + 2x - 6y + 1 + \lambda(2x - y - 2z - 4) = 0$ -----------(1)

But it passes through (0,0,0)

$\therefore$ 0+1+ $\lambda(0-4) = 0 \Rightarrow \lambda = 1/4$ put the value in (1)

$$x^2 + y^2 + z^2 + 2x - 6y + 1 + \frac{1}{4}(2x - y - 2z - 4) = 0$$

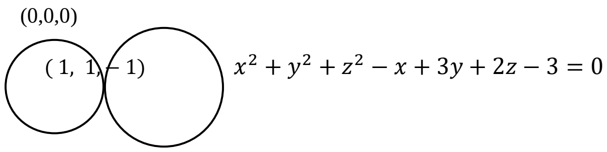
$$\Rightarrow 4(x^2 + y^2 + z^2) + 10x - 25y - 2z = 0$$

6.Find the equation of a sphere which touches the sphere

$$\boldsymbol{x^2 + y^2 + z^2 - x + 3y + 2z - 3 = 0}$$

at the point $(1, 1, -1)$ and passes through origin.

Solution: Let S = $x^2 + y^2 + z^2 - x + 3y + 2z - 3 = 0$



Equation of the tangent plane to the sphere

$S_1 = xx_1 + yy_1 + zz_1 + u(x + x_1) + v(y + y_1) + w(z + z_1) + d = 0.$

P = $(x_1, y_1, z_1) = (1, 1, -1)$

$S_1 = x(1) + y(1) + z(-1) - \frac{1}{2}(x + 1) + \frac{3}{2}(y + 1) + (z - 1) - 3 = 0.$

$$\Rightarrow x + 5y - 6 = 0$$

Equation of the sphere which touches S =0 at P is

$x^2 + y^2 + z^2 - x + 3y + 2z - 3 + \lambda(x + 5y - 6) = 0$ -----------(1)

But it passes through (0,0,0)

$\therefore$ 0 - 3+ $\lambda(0-6) = 0 \Rightarrow \lambda = -1/2$ put the value in (1)

$$x^2 + y^2 + z^2 - x + 3y + 2z - 3 - \frac{1}{2}(x + 5y - 6) = 0$$

$$2(x^2 + y^2 + z^2) - 3x + y + 4z = 0$$

7.Find the equations of tangent planes to the sphere

$x^2 + y^2 + z^2 - 4x + 2y - 6z + 5 = 0$ which are parallel to the plane $2x + 2y - z = 0$.

Solution: Equation of the plane which is parallel to the plane $2x + 2y - z = 0$ is

$$2x + 2y - z + k = 0 \quad \text{---------(1)}$$

But it is tangent to the sphere $x^2 + y^2 + z^2 - 4x + 2y - 6z + 5 = 0$

Centre = (2, -1, 3) and Radius = $\sqrt{4 + 1 + 9 - 5} = \sqrt{9} = 3$

Condition for tangency r = d

$$\Rightarrow\ 3 = \frac{|2(2)+2(-1)-(3)+k\,|}{\sqrt{4+4+1}} = \frac{|-1+k\,|}{3}$$

$\Rightarrow$ 9 = |-1+k |

$\Rightarrow$ -1+k = $\pm$ 9 $\Rightarrow$ k = 10 $or\ -8$

$\therefore$ Equations of tangent planes $2x+2y-z+10=0$ $2x+2y-z-8=0$

8.Find the equations of tangent planes to the sphere

$x^2+y^2+z^2-2x+2y-4z-10=0$ which are parallel to the plane

$x-2y+2z=15$.

Solution: Equation of the plane which is parallel to the plane $x-2y+2z=15$ is

$x-2y+2z+k=0$ ---------(1)

But it is tangent to the sphere $x^2+y^2+z^2-2x+2y-4z-10=0$

Centre = (1, -1, 2) and Radius = $\sqrt{1+1+4+10}$ = $\sqrt{16}$ = 4

Condition for tangency r = d

$$\Rightarrow 4 = \frac{|(1)-2(-1)+2(2)+k|}{\sqrt{4+4+1}} = \frac{|7+k\,|}{3}$$

$\Rightarrow\ 7+k = \pm 12 \Rightarrow k = \pm 12-7 = 5\ or -19$

$\therefore$ Equations of tangent planes $x-2y+2z+5=0$ $x-2y+2z-19=0$

9.Show that the spheres $x^2+y^2+z^2+2x-4y-6z+10=0$

$x^2+y^2+z^2-6x-4y-12z+40=0$ touch externally and find the point of contact.

Solution: Consider $x^2+y^2+z^2+2x-4y-6z+10=0$

Centre A = (-1, 2, 3) and Radius $r_1=\sqrt{1+4+9-10}=2$

Also $x^2+y^2+z^2-6x-4y-12z+40=0$

Centre B = (3, 2, 6) and the radius $r_2=\sqrt{9+4+36-40}=3$

Now AB = $\sqrt{(3+1)^2+(2-2)^2+(6-3)^2}=\sqrt{16+0+9}=5$

$$r_1+r_2=2+3=5=AB$$

$\therefore$ The spheres are touch externally.

(-1, 2, 3) (3, 2, 6)

A r_1 2 : 3 r_2 B

Point of contact = A point and it divides A, B in the ratio 2:3

$$= \left(\frac{6-3}{2+3}, \frac{4+6}{2+3}, \frac{9+12}{2+3}\right) = \left(\frac{3}{5}, \frac{10}{5}, \frac{21}{5}\right)$$

10.Show that the spheres $x^2 + y^2 + z^2 = 25$

$x^2 + y^2 + z^2 - 24x - 40y - 18z + 225 = 0$ touch externally and find the point of contact.

Solution: Consider $x^2 + y^2 + z^2 = 25$

Centre A = (0, 0, 0) and Radius $r_1 = \sqrt{25} = 5$

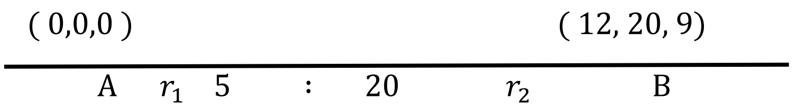
Also $x^2 + y^2 + z^2 - 24x - 40y - 18z + 225 = 0$

Centre B = (12, 20, 9) and the radius $r_2 = \sqrt{144 + 400 + 81 - 225} = \sqrt{400} = 20$

Now AB = $\sqrt{(12-0)^2 + (20-0)^2 + (9-0)^2} = \sqrt{144 + 400 + 81} = \sqrt{625} = 25$

$$r_1 + r_2 = 5 + 20 = 25 = AB$$

∴ The spheres are touch externally.

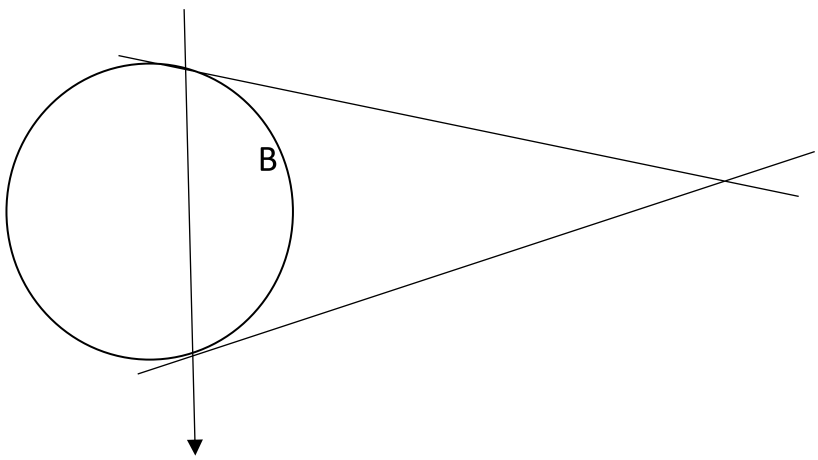


Point of contact = A point and it divides A, B in the ratio 2:3

$$= \left(\frac{0+60}{5+20}, \frac{0+100}{5+20}, \frac{0+45}{5+20}\right) = \left(\frac{12}{5}, 4, \frac{9}{5}\right)$$

Plane of contact, Pole and polar plane

Plane of contact: The locus of points of contact of the tangent planes to the sphere S = 0 which passes through an external point B is called the plane of contact of the point B with respect to the sphere S = 0.

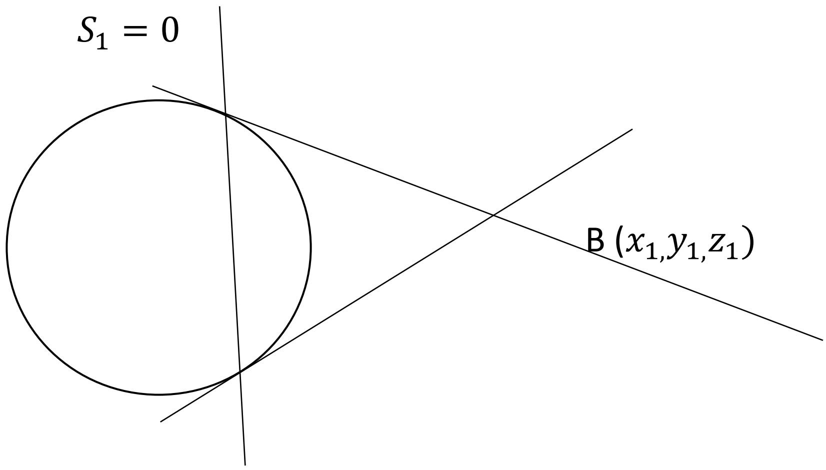


Note: The equation of the plane of contact of the external point B (x_1, y_1, z_1)

w.r.t the sphere S = 0 is $S_1 = 0$.

Pole and polar plane: Let S = 0 be a sphere and B be a given point. The locus of the points, such that the plane of contact of each point w.r.t the sphereS = 0passes through B, is the plane and is called the polar plane of B w.r.t the sphere S = 0. The point B is called pole of the polar plane.

Note: The equation of the polar plane of the point B $(x_{1,}y_{1,}z_1)$w.r.t the sphere S = 0 is $S_1 = 0$.



Formula:

The pole of the plane $lx + my + nz = p \; where \; p \neq 0 \; w.r.t \; the \; sphere$

$x^2 + y^2 + z^2 = r^2$ is $\left(\frac{lr^2}{p}, \frac{mr^2}{p}, \frac{nr^2}{p},\right)$.

Conjugate points: The points A $(x_{1,}y_{1,}z_1)$ $B\ (x_{2,}y_{2,}z_2)$are said to be conjugate w.r.t the sphere S = 0 is the polar of A passes through Condition for conjugate points $S_{12} = 0$.

Conjugate planes: Two planes $\pi_1 = 0 \; and \; \pi_2 = 0$are said to be conjugate planes w.r.t S = 0 if the pole of $\pi_1 = 0 \; lies \; on \; \pi_2 = 0$

Note: If the planes $l_1 x + m_1 y + n_1 z = p_1$ and $l_2 x + m_2 y + n_2 z = p_2$
Are conjugate w.r.t the sphere $x^2 + y^2 + z^2 = r^2$ then

$$r^2(l_1 l_2 + m_1 m_2 + n_1 n_2) = p_1 p_2.$$

Problems:

1.Find the pole of the plane $x - y + 5z - 3 = 0 \; w.r.t \; the \; sphere \; x^2 + y^2 + z^2 = 9$

Solution: Let P = $(x_{1,}y_{1,}\, z_1)$is pole of the plane

$$x - y + 5z - 3 = 0 \quad \text{------------(1)}$$

Equation of polar of P = $(x_{1,}y_{1,}z_1) w.r.t \; the \; sphere$

$$x^2 + y^2 + z^2 = 9\ is$$

$$xx_1 + yy_1 + zz_1 - 9 = 0 \text{ --------(2)}$$

From (1) and (2) $\frac{x_1}{1} = \frac{y_1}{-1} = \frac{z_1}{5} = \frac{9}{3} = 3 \Rightarrow x_1 = 3, y_1 = -3, z_1 = 15$

Pole of the plane = (3,-3,15)

2.Show that the points (1, -1,2)(5,2,3)are conjugate w.r.t the sphere $x^2 + y^2 + z^2 = 9$.

Solution: Condition for conjugate planes $S_{12} = 0$

Given A $(x_1, y_1, z_1) = (1,-1,2)$, $B\ (x_2, y_2, z_2) = (5,2,3)$

Now $S_{12} = x_1x_2 + y_1y_2 + z_1z_2 - 9$

$$= (1)(5) + (-1)(2) + (2)(3) - 9 = 5 - 2 + 6 - 9 = 0$$

Therefore, points are conjugate.

3.Show that the planes $5x - y - 6z + 25 = 0$ $x - 2y - 3z + 25 = 0$ are conjugate w.r.t the sphere $x^2 + y^2 + z^2 = 25$.

Solution: Condition for conjugate planes

$$r^2(l_1l_2 + m_1m_2 + n_1n_2) = p_1p_2.$$

LHS = $r^2(l_1l_2 + m_1m_2 + n_1n_2) = 25\,[\,5(1) + (-1)(-2) + (-6)(-3)]$

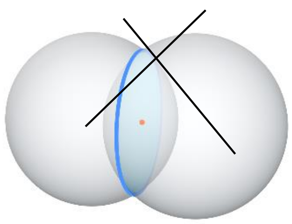
= 25 (5 + 2 + 18) = 625

RHS = p_1p_2 = $25 \times 25 = 625$∴ LHS = RHS

⇒ The planes are conjugate.

Angle of Intersection of two spheres

1.Angle between two intersecting spheres is the angle made by the tangent planes of the spheres at the point of intersection.



2.If d is the distance between the centres of two intersecting spheres and r_1,r_2

are radii of the spheres then angle between the spheres

$$\cos\theta = \pm \frac{{r_1}^2 + {r_2}^2 - d^2}{2r_1r_2}$$

3.If the spheres are intersecting orthogonally then $\theta = 90^0$

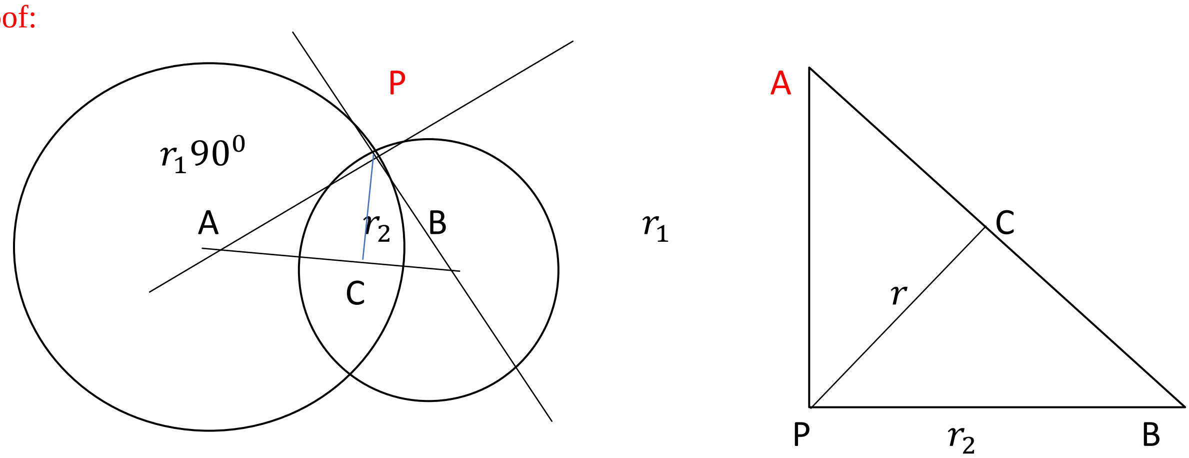
then ${r_1}^2 + {r_2}^2 = d^2$

4.If $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ *and* $x^2 + y^2 + z^2 + 2u'x + 2v'y + 2w'z + d' = 0$ *are cut orthoganally then*

$$2uu' + 2vv' + 2ww' = d + d'$$

4.Theorem: If r_1,r_2 are radii of the two orthogonal spheres, then prove that the radius of the circle of their intersection is $\frac{r_1r_2}{\sqrt{{r_1}^2 + {r_2}^2}}$

Proof:



In the above diagram they are two intersection spheres which cuts orthogonally at P.

Let A and B are centres of the spheres and CP is the radius of the circle.

AP = Radius of first sphere = r_1 PB = Radius of the second sphere = r_2

Let PC =Radius of the circle = r

In the above diagram there are 3 Right angle tringles $\Delta APB, \Delta ACP$ *and* ΔBCP

In the right-angle triangle APB $AB^2 = AP^2 + BP^2$

$(AC + CB)^2 = {r_1}^2 + {r_2}^2$ -----------(1)

$AC^2 + CB^2 + 2AC.CB = {r_1}^2 + {r_2}^2$

$(AP^2 - CP^2) + (BP^2 - CP^2) + 2AC.CB = {r_1}^2 + {r_2}^2$

$$({r_1}^2 - r^2) + ({r_2}^2 - r^2) + 2AC.CB = {r_1}^2 + {r_2}^2$$

$$\Rightarrow -2r^2 + 2AC.CB = 0$$

$$\Rightarrow -2r^2 = -2AC.CB$$

$$\Rightarrow \; r^2 = AC.CB$$

Squaring on both sides

$$\Rightarrow r^4 = AC^2 \; CB^2 = ({r_1}^2 - r^2)({r_2}^2 - r^2)$$

$$\Rightarrow r^4 = {r_1}^2{r_1}^2 - {r_1}^2r^2 - r^2{r_1}^2 + r^4$$

$$\Rightarrow {r_1}^2{r_1}^2 - {r_1}^2r^2 - r^2{r_1}^2 = 0$$

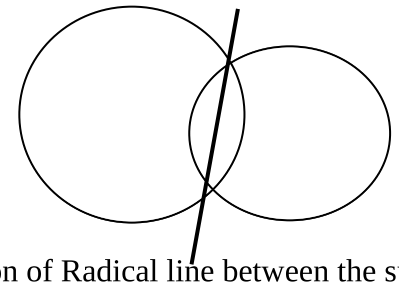
$$\Rightarrow {r_1}^2{r_1}^2 = ({r_1}^2 + {r_1}^2)r^2$$

$$\Rightarrow \; r^2 = \frac{{r_1}^2{r_1}^2}{{r_1}^2+{r_1}^2}$$

$$\Rightarrow \; Radius\ of\ the\ circle\ r = \frac{r_1r_2}{\sqrt{{r_1}^2+{r_2}^2}}$$

Radical plane:

The equation of Radical plane between the plane S=0 and S' = 0 is S - S' = 0



Radical line: The equation of Radical line between the spheres $S = 0,\ S'$=0 and$S'' = 0$ is if $\pi = S - S' = 0\ and\ \pi' = S - S'' = 0$ then $\pi = 0 = \pi'$

Radical centre: The radical centre between the spheres $S = 0, S'$=0

$S'' = 0\ and\ S''' = 0$ is the intersection point of radical lines

$$\pi = S - S' = 0\ and\ \pi' = S' - S'' = 0\ \ \pi'' = S'' - S''' = 0$$

$\pi''' = S''' - S'''' = 0$ then $\pi = 0 = \pi'$then $\pi'' = 0 = \pi'''$

Problems:

1.Find the radical plane of the spheres

$$x^2 + y^2 + z^2 + 4x - 2y + 2z + 6 = 0,\ x^2 + y^2 + z^2 + 2x - 4y - 2z + 6 = 0$$

Solution: Equation of radical plane between the plane S = 0 and S' = 0 is S- S' = 0

$$\therefore\ (x^2 + y^2 + z^2 + 4x - 2y + 2z + 6) - (x^2 + y^2 + z^2 + 2x - 4y - 2z + 6) = 0$$

$$2x + 2y + 4z = 0 \Rightarrow x + y + 2z = 0$$

2.Show that the spheres

$$x^2 + y^2 + z^2 + 6y + 2z + 8 = 0,\ x^2 + y^2 + z^2 + 6x + 8y + 4z + 20 = 0$$

are orthogonal.

Solution: If $x^2+y^2+z^2+2ux+2vy+2wz+d=0$

$and\ \ x^2+y^2+z^2+2u'x+2v'y+2w'z+d'=0\ are\ cut\ orthoganally\ then$

$$2uu'+2vv'+2ww' = d+d'$$

$$x^2+y^2+z^2+6y+2z+8=0$$

$$2u=0 \qquad 2v=6,\quad 2w=2 \qquad d=8$$

$$x^2+y^2+z^2+6x+8y+4z+20=0$$

$$2u'=6 \qquad 2v'=8 \qquad 2w'=4 \qquad d'=20$$

$$\Rightarrow u'=3 \qquad v'=4 \qquad w'=2 \qquad d'=20$$

LHS = $2uu'+2vv'+2ww'=(0)(3)+(6)(4)+(2)(2)=28=8+20=d+d'=RHS$

3.Find the equation of a sphere through a circle

$x^2+y^2+z^2-2x+3y-4z+6=0, 3x-4y+5z-15=0$ and

cutting the sphere $x^2+y^2+z^2+2x+4y-6z+11=0$ orthogonally.

Solution:

Equation of the circle through the circle S = 0 = π is S + $\lambda\,\pi=0\ where\ \lambda\ is\ a\ constant$

$$\therefore x^2+y^2+z^2-2x+3y-4z+6+\lambda(\,3x-4y+5z-15)=0\text{---------}(1)$$

$$x^2+y^2+z^2-(2-3\lambda)x+(3-4\lambda)y-(4-5\lambda)z+(6-15\,\lambda)=0$$

Let $2u=\ -(2-3\lambda),\ 2v=(3-4\lambda),\ 2w=-(4-5\lambda),\ and\ d=\ 6-15\,\lambda$

But it is orthogonal to the sphere $x^2+y^2+z^2+2x+4y-6z+11=0$

$$2u'=2, \qquad 2v'=4, \qquad 2w'=-6 \qquad and \quad d'=1$$

$$\Rightarrow u'=1, \qquad v'=2, \qquad w'=-3 \qquad and \quad d'=11$$

Condition for orthogonal spheres

$2uu'+2vv'+2ww' = d+d'$

$$-(2-3\lambda)(1)+\ (3-4\lambda)(2)-\ (4-5\lambda)(-3)=\ 6-15\,\lambda+11$$

$$\Rightarrow(-2+3\lambda)+\ (6-8\lambda)+\ (12-15\lambda)=\ 17-15\,\lambda$$

$$\Rightarrow 16-20\lambda\ =17-15\lambda\ \ \Rightarrow -5\lambda=1 \Rightarrow\ \lambda=-\frac{1}{5}$$

Put the value in (1)

$$x^2+y^2+z^2-2x+3y-4z+6-\frac{1}{5}(\,3x-4y+5z-15)=0$$

$5(\,x^2+y^2+z^2)-13x+19y-25z+45=0.$

4. Find the equation of a sphere through a circle

$x^2+y^2+z^2+3x+y+2z-2=0, x+3y-2z+1=0$ and

cutting the sphere $x^2+y^2+z^2+x-3z-2=0$orthogonally.

Solution:

Equation of the circle through the circle S = 0 = π is S + $\lambda\,\pi = 0$ *where λ is a constant*

$\therefore x^2+y^2+z^2+3x+y+2z-2+\lambda(x+3y-2z+1)=0$

$x^2+y^2+z^2+(3+\lambda)x+(1+3\lambda)y+(2-2\lambda)z+(-2+\lambda)=0$---------(1)

Let $2u=(3+\lambda),\ 2v=(1+3\lambda),\ 2w=(2-2\lambda),\ and\ d=-2+\lambda$

But it is orthogonal to the sphere $x^2+y^2+z^2+x-3z-2=0$

$$2u'=1,\qquad 2v'=0,\qquad 2w'=-3\quad and\quad d'=-2$$

$$\Rightarrow u'=\frac{1}{2},\qquad v'=0,\qquad w'=-\frac{3}{2}\quad and\quad d'=-2$$

Condition for orthogonal spheres

$2uu'+2vv'+2ww'=d+d'$

$$(3+\lambda)\left(\frac{1}{2}\right)+(1+3\lambda)(0)+(2-2\lambda)\left(-\frac{3}{2}\right)=-2+\lambda-2$$

$\Rightarrow\ (3+\lambda)-6+6\lambda=2(-4+\lambda)$

$$\Rightarrow 7\lambda-3=-8+2\lambda\ \Rightarrow\ 5\lambda=-5\ \Rightarrow\ \lambda=-1$$

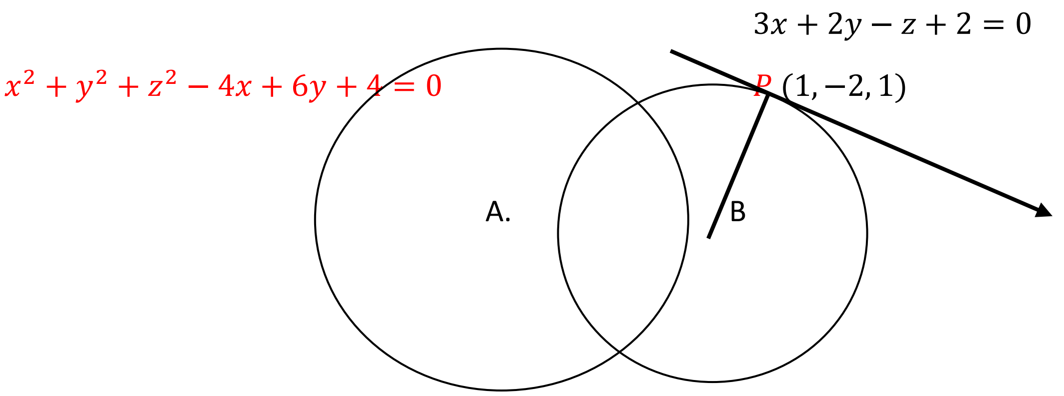
Put the value in (1)

$$x^2+y^2+z^2+(3-1)x+(1+3(-`1))y+(2-2(-1))z+(-2+(-1))=0$$

$$x^2+y^2+z^2+2x-2y+4z-3=0$$

5.Find the equation of a sphere which touches the plane $3x+2y-z+2=0$ at the point $(1,-2,1)$ and cuts orthogonally the sphere $x^2+y^2+z^2-4x+6y+4=0$.

Solution:



To solve this problem using the orthogonal condition

${r_1}^2 + {r_2}^2 = d^2 \quad where\ d = distence\ between\ the\ centres\ of\ the\ spheres$.

Consider the given sphere $x^2 + y^2 + z^2 - 4x + 6y + 4 = 0.$

Centre A = (2, -3, 0) and radius $r_1 = \sqrt{4 + 9 + 0 - 4} = \sqrt{9} = 3$

Now equation of tangent plane $3x + 2y - z + 2 = 0$

Drs of the normal line of the plane = drs. of the line BP = 3,2,-1

Where P = (1, -2, 1) and the centre of the required sphere = B

Equation of the line BP is

$$\frac{x-1}{3} = \frac{y+2}{2} = \frac{z-1}{-1} = t\ say$$

$$\Rightarrow x = 3t + 1, y = 2t - 2, z = -t + 1$$

B = $(3t + 1, 2t - 2, -t + 1)$

Drs of the BP = $3t + 1 - 1, 2t - 2 - 2, -t + 1 - 1 = 3t, 2t, -t$

$\therefore r_2 = BP = \sqrt{(3t)^2 + (2t)^2 + t^2}\sqrt{9t^2 + 4t^2 + t^2} = \sqrt{14t^2}$

$d = distence\ between\ the\ centres\ A = (2, -3, 0)\ P = (3t + 1, 2t - 2, -t + 1)$

$$= \sqrt{3t + 1 - 2)^2 + (2t - 2 + 3)^2 + (t + 1 - 0)^2}$$

$$= \sqrt{3t - 1)^2 + (2t + 1)^2 + (t + 1)^2}$$

$$= \sqrt{9t^2 - 6t + 1 + 4t^2 + 4t + 1 + t^2 - 2t + 1}$$

$$= \sqrt{14t^2 - 4t + 3}$$

Now $\quad {r_1}^2 + {r_2}^2 = d^2$

$$3^2 + [\sqrt{14t^2}\,]^2 = \left[\sqrt{14t^2 - 4t + 3}\right]^2$$

$$9 + 14t^2 = 14t^2 - 4t + 3$$

$$\Rightarrow 6 = -4t \ \Rightarrow t = -\frac{3}{2}$$

Centre B =$(3t + 1, 2t - 2, -t + 1)$= $(3(-\frac{3}{2}) + 1, 2(-\frac{3}{2}) - 2, -(-\frac{3}{2}) + 1) = (-\frac{7}{2}, -\frac{10}{2}, \frac{5}{2})$

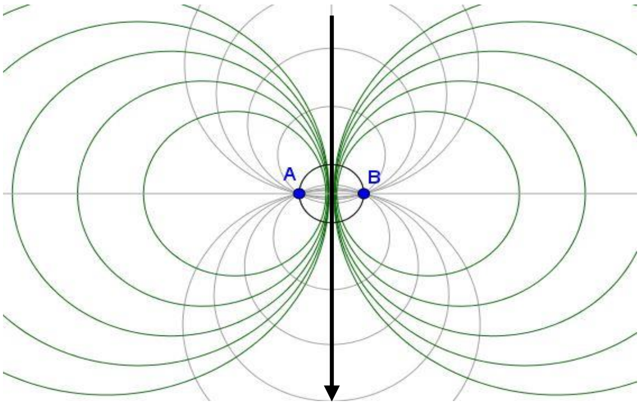
Radius $\therefore r_2 = \sqrt{14t^2} = \sqrt{14(-\frac{3}{2})^2}$ = $\sqrt{126/4} = \sqrt{63/2}$

Equation of the sphere centre at $B = \left(-\frac{7}{2}, -\frac{10}{2}, \frac{5}{2}\right)\ radius = \sqrt{63/2}$

$$(x + \frac{7}{2})^2 + (y + \frac{10}{2})^2 + (z - \frac{5}{2})^2 = \frac{63}{2}$$

COAXAL SYSTEM OF SPHERES

Definition: A system of spheres is said to be a coaxal system of spheres if any two spheres of the system have the same radical plane.



1.Equation of the coaxal system of spheres S= 0 is a member and radical plane $\pi = 0$is

$$S + \lambda\pi = 0.$$

2. The centres of all point spheres(radius = 0) of the coaxal system are the limiting points of the coaxal system.

Problems:

1.Find the limiting points of the coaxal system of spheres

$$\boldsymbol{x^2 + y^2 + z^2 - 20x + 30y - 40z + 29 + \lambda(2x - 3y + 4z) = 0}$$

Solution: Given system of sphere

$$x^2 + y^2 + z^2 - 20x + 30y - 40z + 29 + \lambda(2x - 3y + 4z) = 0$$

$$x^2 + y^2 + z^2 - (20 - 2\lambda)x + (30 - 3\lambda)y - (40 - 4\lambda)z + 29 = 0$$

$$Centre\ of\ the\ sphere = \left(\frac{20-2\lambda}{2}, \frac{-(30-3\lambda)}{2}, \frac{40-4\lambda}{2}\right) = \left(\frac{2(10-\lambda)}{2}, \frac{-3(10-\lambda)}{2}, \frac{4(10-\lambda)}{2}\right)$$

$$Radius = \sqrt{\frac{4(10-\lambda)^2}{4} + \frac{9(10-\lambda)^2}{4} + \frac{16(10-\lambda)^2}{4} - 29}$$

$$= \sqrt{\frac{29(10-\lambda)^2}{4} - 29}$$

$$To\ find\ the\ limiting\ points\ put\ the\ radius = 0 \Rightarrow\ r^2 = 0$$

$$\therefore \frac{29(10-\lambda)^2}{4} - 29 = 0 \Rightarrow \frac{(10-\lambda)^2}{4} - 1 = 0$$

$$\Rightarrow \frac{(10-\lambda)^2}{4} = 1\ \Rightarrow\ (10-\lambda)^2 = 4$$

$$\Rightarrow 10-\lambda=\pm 2 \;\Rightarrow\; \lambda=\pm 2+10$$

$$\lambda=8,12$$

Put $\lambda=8$ in the centre$=\left(\frac{2(10-\lambda)}{2},\frac{-3(10-\lambda)}{2},\frac{4(10-\lambda)}{2}\right)$

$$=\left(\frac{2(10-8)}{2},\frac{-3(10-8)}{2},\frac{4(10-8)}{2}\right)=(2,-3,4)$$

$$\lambda=12\ we\ get\ \left(\frac{2(10-12)}{2},\frac{-3(10-12)}{2},\frac{4(10-12)}{2}\right)=(-2,3,-4)$$

$\therefore$ The required limiting points $=(2,-3,4)\ (-2,3,-4)$

2.Find the limiting points of the coaxal system of spheres determined by the spheres

$$\boldsymbol{x^2+y^2+z^2+3x-3y+6=0\ \ x^2+y^2+z^2-6y-6z+6=0}$$

Solution: Given system of sphere

$$x^2+y^2+z^2+3x-3y+6=0\ \ x^2+y^2+z^2-6y-6z+6=0$$

Equation of radical plane

$$\pi=\ x^2+y^2+z^2+3x-3y+6-(\ x^2+y^2+z^2-6y-6z+6)=0$$

$= 3x+3y+6z=0 \Rightarrow x+y+2z=0 \Rightarrow 2x+2y+4z=0$

Equation of coaxal system

$$x^2+y^2+z^2-6y-6z+6+\lambda(2x+2y+4z)=0$$

$$x^2+y^2+z^2+2\lambda x+(2\lambda-6)y+(4\lambda-6)z+6=0$$

$Centre\ of\ the\ sphere=\left(\frac{-2\lambda}{2},\frac{-(2\lambda-6)}{2},\frac{-(4\lambda-6}{2}\right)=(-\lambda,-(\lambda-3),-(2\lambda-3)\ )$

$$Radius=\sqrt{\lambda^2+(\lambda-3)^2+(2\lambda-3)^2-6}$$

$$To\ find\ the\ limiting\ points\ put\ the\ radius=0\Rightarrow\ r^2=0$$

$$\lambda^2+(\lambda-3)^2+(2\lambda-3)^2-6$$

$\Rightarrow\lambda^2+\lambda^2-6\lambda+9+4\lambda^2-12\lambda+9-6=0$

$\Rightarrow 6\lambda^2$-$18\lambda+12=0\Rightarrow\lambda^2-3\lambda+2=0\Rightarrow\ \lambda=1\ or\ 2$

Put $\lambda=1\ in\ the\ centre\ (-\lambda,-(\lambda-3),-(2\lambda-3)\ )$$=(-1,-(1-3),-(2(1)-3)\ )$

$=$(-1, 2, 1)

Put $\lambda=2\ in\ the\ centre\ (-2,-(2-3),-(2(2)-3)=(-2,1,-1)$

The required limiting points = (-1, 2, 1) (-2, 1, -1)

3.Find the limiting points of the coaxal system of spheres determined by the spheres

$$x^2 + y^2 + z^2 + 3x - 3y + 6 = 0\ ,\ x^2 + y^2 + z^2 + 4x - 2y + 2z + 6 = 0$$

Solution: Given system of sphere

$$x^2 + y^2 + z^2 + 3x - 3y + 6 = 0,\ x^2 + y^2 + z^2 + 4x - 2y + 2z + 6 = 0$$

Equation of radical plane

$$\pi =\ x^2 + y^2 + z^2 + 3x - 3y + 6 - (\ x^2 + y^2 + z^2 + 4x - 2y + 2z + 6 = 0\ ) = 0$$

$$\Rightarrow \pi = x + y + 2z = 0$$

Equation of coaxal system

$$x^2 + y^2 + z^2 + 3x - 3y + 6 + \lambda(x + y + 2z) = 0$$

$$x^2 + y^2 + z^2 + (3 + \lambda)x + (-3 + \lambda)y + 2\lambda z + 6 = 0$$

$$Centre\ of\ the\ sphere = \left(\frac{-(3+\lambda)}{2}, \frac{-(-3+\lambda)}{2}, \frac{-2\lambda}{2}\right)$$

$$To\ find\ the\ limiting\ points\ put\ the\ radius = 0 \Rightarrow\ r^2 = 0$$

$$\frac{(3+\lambda)^2}{4} + \frac{(-3+\lambda)^2}{4} + \frac{4\lambda^2}{4} - 6 = 0$$

$$\Rightarrow \lambda^2 + 6\lambda + 9 + \lambda^2 - 6\lambda + 9 + 4\lambda^2 - 24 = 0$$

$$\Rightarrow 6\lambda^2 - 6 = 0 \Rightarrow \lambda^2 = 1 \Rightarrow\ \lambda = \pm 1$$

Put $\lambda = 1\ in\ the\ centre\ \left(\frac{-(3+\lambda)}{2}, \frac{-(-3+\lambda)}{2}, \frac{-2\lambda}{2}\right)$

$$= \left(\frac{-(3+1)}{2}, \frac{-(-3+1)}{2}, \frac{-2(1)}{2}\right) = (-2, 1, -1)$$

Put $\lambda = -1\ in\ the\ centre = \left(\frac{-(3-1)}{2}, \frac{-(-3-1)}{2}, \frac{-2(-1)}{2}\right) = (-1, 2, 1)$

The required limiting points $=(-2, 1, -1)\ (-1, 2, 1)$.

4.Find the limiting points of the coaxal system of spheres determined by the spheres

$$x^2 + y^2 + z^2 + 4x - 2y + 2z + 6 = 0,\ x^2 + y^2 + z^2 + 2x - 4y - 2z + 6 = 0$$

Solution: Given system of sphere

$$x^2 + y^2 + z^2 + 4x - 2y + 2z + 6 = 0,\ x^2 + y^2 + z^2 + 2x - 4y - 2z + 6 = 0$$

Equation of radical plane

$$\pi =\ x^2 + y^2 + z^2 + 4x - 2y + 2z + 6 - (\ x^2 + y^2 + z^2 + 2x - 4y - 2z + 6) = 0$$

$$\Rightarrow \pi = 2x + 2y + 4z = 0$$

Equation of coaxal system

$$x^2 + y^2 + z^2 + 4x - 2y + 2z + 6 + \lambda(2x + 2y + 4z) = 0$$

$$x^2+y^2+z^2+(4+2\lambda)x+(-2+2\lambda)y+(2+4\lambda)z+6=0$$

$$Centre\ of\ the\ sphere=\left(\frac{-(4+2\lambda)}{2},\frac{-(-2+2\lambda)}{2},\frac{-(2+4\lambda}{2}\right)$$

$$=(-(2+\lambda),(1-\lambda),-(1+2\lambda))$$

$$To\ find\ the\ limiting\ points\ put\ the\ radius=0\Rightarrow\ r^2=0$$

$$(2+\lambda)^2\ +(1-\lambda)^2\ +\ (1+2\lambda)^2$$

$$\Rightarrow\ \lambda^2+4\lambda+4+\lambda^2-2\lambda+1+1+4\lambda^2+4\lambda-6=0$$

$$\Rightarrow 6\lambda^2+6\lambda=0\Rightarrow\lambda(\lambda+1)=0\ \ \lambda=0,-1$$

Put $\lambda=0\ in\ the\ centre\ =(-(2+\lambda),(1-\lambda),-(1+2\lambda))$

$$==(-(2+0),(1-0),-(1+0)\ )\ =(-2,1,-1)$$

Put $\lambda=-1\ in\ the\ centre==(-(2-1),(1+1),-(1-2))\ =(-1,2,1)$

The required limiting points $=(-2,1,-1)\ (-1,2,1)$.

5. Find the radical plane of the coaxal system whose limiting points are (3, 1, -2)(5,-3,4).

Solution: Equations of point spheres whose centres are (3,1,-2)(5,-3,4)

$(x-3)^2+(y-1)^2+(z+2)^2=0$

$\Rightarrow x^2-6x+9+y^2-2y+1+z^2+4z+4=0$

$S=\ x^2+y^2+z^2-6x-2y+4z+14=0$

Equations of point spheres whose centres are (5, -3, 4)

$(x-5)^2+(y+3)^2+(z-4)^2=0$

$\Rightarrow x^2-10x+25+y^2+6y+9+z^2-8z+16=0$

$S'=x^2+y^2+z^2-10x+6y-8z+50=0$

Equation of radical plane $\pi=S-S'=0$

$\Rightarrow x^2+y^2+z^2-6x-2y+4z+14-(\ x^2+y^2+z^2-10x+6y-8z+50)=0$

$\Rightarrow 44x-8y+12z-36=0\Rightarrow x-2y+3z-9=0$

All the Best

GOVERNMENT DEGREE COLLEGE, RAVULAPALEM

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THE SPHERE -NOTES

FIRST B.SC MATHEMATICS

SEMESTER -II

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LECTURER IN MATHEMATICS

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The cone

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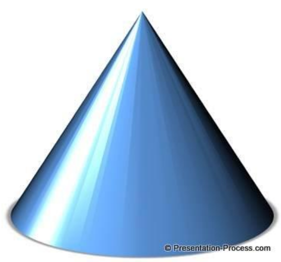
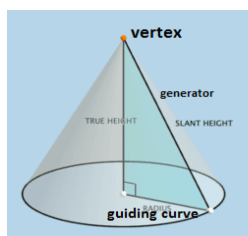
Definition: A cone is a surface generated by a straight line which passes through a fixed point and intersecting a given curve or touches a given surface. The fixed point is called the vertices of the cone and the given curve (surface) is called the guiding curve or surface of the cone and the straight line is called generator of the cone.





Note: Every homogeneous equation of nth degree in x, y and z represents a cone whose vertices at origin.

1. The general equation of the cone whose vertices at origin is in the form

$$ax^2 + by^2 + cz^2 + 2hxy + 2fyz + 2gzx = 0$$

2. A line $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ $is\ a\ generator\ of\ the\ cone$

$$ax^2 + by^2 + cz^2 + 2hxy + 2fyz + 2gzx = 0$$

if and only if $al^2 + bm^2 + cn^2 + 2hlm + 2fmn + 2gnl = 0.$

Problems: Show that $\frac{x}{1} = \frac{y}{-1} = \frac{z}{-1}$ *is a generator of the cone* $5yz + 8zx - 3xy = 0$.

Solution: Put x = 1, y = -1, z = -1 in $5yz + 8zx - 3xy = 0$

LHS = $5yz + 8zx - 3xy = 0 = 5(-1)(-1) + 8(-1)(1) - 3(1)(-1) = 5 - 8 + 3 = 0\ RHS$

$\therefore\ \frac{x}{1} = \frac{y}{-1} = \frac{z}{-1}$ *is a generator of the cone* $5yz + 8zx - 3xy = 0$

Result: Show that the general equation of the cone of second degree which passes through the coordinate axes is $hxy + fyz + gzx = 0$.

Proof: We the equation of the cone of second order is

$$ax^2 + by^2 + cz^2 + 2hxy + 2fyz + 2gzx = 0 \quad \text{----------(1)}$$

If X-axis is the generator of the cone (1)

We know the equation of X- axis $\frac{x}{1} = \frac{y}{0} = \frac{z}{0}$ and is the generator of (1)

$$\therefore \quad a(1)^2 + b(0)^2 + c(0)^2 + 2h(1)(0) + 2f(0)(0) + 2g(0)(1) = 0$$

$\Rightarrow$ a = 0.

Similarly, another two axes $Y - axis\ \frac{x}{0} = \frac{y}{1} = \frac{z}{0}$ *and* $Z - axis\ \frac{x}{0} = \frac{y}{0} = \frac{z}{1}$ are generators of the cone (1)

We get b = 0 & c = 0

Put the values in (1) $2hxy + 2fyz + 2gzx = 0 \Rightarrow hxy + fyz + gzx = 0$

Model-1

Problem-1 Find the equation of the cone which passes through the coordinate axes and the lines

$$\frac{x}{1} = \frac{y}{-2} = \frac{z}{3} \text{ and } \frac{x}{3} = \frac{y}{-1} = \frac{z}{1}.$$

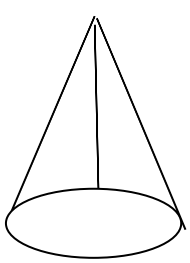
Solution: We know the equation of the cone passing through the coordinate axes is

$$hxy + fyz + gzx = 0 \quad \text{---------(1)}$$

If $\frac{x}{1} = \frac{y}{-2} = \frac{z}{3}$ is a generator of the cone (1)

$$h(-2) + f(-6) + g(3) = 0 \quad \text{---------(2)}$$

and $\frac{x}{3} = \frac{y}{-1} = \frac{z}{1}$ is also generator of the cone (1)



$$h(-3) + f(-1) + g(3) = 0 \quad \text{----------(3)}$$

To solve the equations (1) & (2)

-6	3	-2	-6
-1	3	-3	-1

$$\frac{h}{-18+3} = \frac{f}{-9+6} = \frac{g}{2-18} \Rightarrow \frac{h}{-15} = \frac{f}{-3} = \frac{g}{-16} \Rightarrow \frac{h}{15} = \frac{f}{3} = \frac{g}{16}$$

Hence the equation of required cone $15xy + 3yz + 16zx = 0$

Problem-2 Find the equation of the cone which passes through the coordinate axes and the lines

$$\frac{x}{3} = \frac{y}{5} = \frac{z}{1} \; and \; \frac{x}{1} = \frac{y}{-1} = \frac{z}{2}.\frac{x}{-11} = \frac{y}{5} = \frac{z}{8}$$

Solution: We know the equation of the cone passing through the coordinate axes is

$$hxy + fyz + gzx = 0 \quad \text{---------(1)}$$

If $\frac{x}{3} = \frac{y}{5} = \frac{z}{1}$ is a generator of the cone (1)

$$h(15) + f(5) + g(3) = 0 \quad \text{---------(2)}$$

and $\frac{x}{1} = \frac{y}{-1} = \frac{z}{2}$ is also generator of the cone (1)

$$h(-1) + f(-2) + g(2) = 0 \quad \text{----------(3)}$$

To solve the equations (1) & (2)

5	3	15	5
-2	2	-1	-2

$$\frac{h}{10+6} = \frac{f}{-3-30} = \frac{g}{-30+5} \Rightarrow \frac{h}{16} = \frac{f}{-33} = \frac{g}{-25} \Rightarrow \frac{h}{16} = \frac{f}{-33} = \frac{g}{-25}$$

Hence the equation of required cone $16xy - 33yz - 25zx = 0$

If you put the drs of the line $\frac{x}{-11} = \frac{y}{5} = \frac{z}{8}$ in the above equation

LHS = $16xy - 33yz - 25zx = 16(-11)(5) - 33(5)(8) - 25(8)(-11)$

$$= -880 - 1320 + 2200 = 0 = RHS$$

Model-2

Problem :3 Find the equation of the cone with vertex at origin and whose base curve is

$x^2 + y^2 = 4, z = 2$

Solution: Equation of the line through (0,0,0) is $\frac{x}{l} = \frac{y}{m} = \frac{z}{n} = r\ say$---------(1)

$\Rightarrow x = lr, y = mr, z = nr$

$\therefore$ Any point on the line P= (lr, mr, nr)

If the point lies on the curve $x^2 + y^2 = 4, z = 2$

We get $(lr)^2 + (mr)^2 = 4,\ nr = 2\ \Rightarrow r^2(l^2 + m^2) = 4,\ r = \frac{2}{n}$

Put the value $r\ in\ \ r^2(l^2 + m^2) = 4 \Rightarrow (\frac{2}{n})^2(l^2 + m^2) = 4$

$\Rightarrow (l^2 + m^2) = n^2$ and using (1) to eliminate l, m, n

We get $(x^2 + y^2) = z^2$ and is the equation of the cone.

Problem :4

Find the equation of the cone whose vertex is (1,1,0) and whose base curve is

$x^2 + z^2 = 4, y = 0$

Solution: Equation of the line through (1, 1, 0) is $\frac{x-1}{l} = \frac{y-1}{m} = \frac{z-0}{n} = r\ say$---------(1)

$\Rightarrow x = lr + 1, y = mr + 1, z = nr$

$\therefore$ Any point on the line P= $(lr + 1,\ mr + 1,\ nr)$

If the point lies on the curve $x^2 + z^2 = 4, y = 0$

We get $(lr + 1)^2 + (nr)^2 = 4,\ mr + 1 = 0\ \Rightarrow (lr + 1)^2 + (nr)^2 = 4,\ r = \frac{-1}{m}$

Put the value $r\ in\ \ (lr + 1)^2 + n^2r^2) = 4 \Rightarrow \left(\frac{-l}{m} + 1)^2 + n^2[\frac{-1}{m}]^2\right) = 4$

$(m - l)^2 + n^2 = 4m^2$ and using (1) to eliminate l, m

$\Rightarrow [(y - 1) - (x - 1)]^2 + z^2 = 4(y - 1)^2$

We get $x^2 + 3y^2 + z^2 - 2xy + 8y - 4 = 0$ and is the equation of the cone.

Problem: 5

Find the equation of the cone whose vertex is (5, 4, 3) and whose base curve is

$3x^2 + 2y^2 = 6, \ y + z = 0$

Solution: Equation of the line through (5, 4, 3) is $\frac{x-5}{l} = \frac{y-4}{m} = \frac{z-3}{n} = r\ say$---------(1)

$\Rightarrow x = lr + 5, y = mr + 4, z = nr + 3$

∴ Any point on the line P= ($lr + 5, \ mr + 4, \ nr + 3$)

If the point lies on the curve $3x^2 + 2y^2 = 6, \ y + z = 0$

We get $3(lr + 5)^2 + 2(mr + 4)^2 = 6,$

$and \ \ mr + 4 + nr + 3 = 0 \Rightarrow r(m + n) = -7 \ \Rightarrow r = \frac{-7}{m+n}$

Now to eliminate r in $3(lr + 5)^2 + 2(mr + 4)^2 = 6,$

$$3\left(\frac{-7l}{m+n} + 5\right)^2 + 2\left(\frac{-7m}{m+n} + 4\right)^2 = 6$$

$3(5m + 5n - 7l)^2 + 2(4m + 4n - 7m)^2 = 6(m + n)^2$

and using (1) to eliminate l, m, n

$3([5(y-4)) + 5(z-3) - 7(x-5)])^2 + 2([4(y-4) + 4(z-3) - 7(y-4)])^2 = 6[(y-4) + (z-3)]^2$

$$3(-7x + 5y + 5z)^2 + 2(-3y + 4z)^2 = 6(y + z - 7)^2$$

$\Rightarrow$ We get $147x^2 + 87y^2 + 101z^2 - 210xy + 90yz - 210zx - 294. = 0$ and is the equation of the cone.

Problem: 6. Find the equation of the cone whose vertex is (1, 2, 3) and whose base curve is

$$y^2 = 4ax, z = 0$$

Solution: Equation of the line through (1, 2, 3) is $\frac{x-1}{l} = \frac{y-2}{m} = \frac{z-3}{n} = r\ say$---------(1)

$\Rightarrow x = lr + 1, y = mr + 2, z = nr + 3$

∴ Any point on the line P= ($lr + 1, \ mr + 2, \ nr + 3$)

If the point lies on the curve $y^2 = 4ax, z = 0$

We $(mr+2)^2 = 4a(lr+1),\ \ nr+3=0\ \Rightarrow r = \frac{-3}{n}$

Put the value $r\ in\ \ (\frac{-3m}{n}+2)^2 = 4a\left(\frac{-3l}{n}+1\right)$

$(2n-3m)^2 = 4a(n-3l)n$

Now from (1) to eliminate r in the above equation

$(2(z-3)-3(y-2))^2 = 4a\big((z-3)-3(x-1)\big)(z-3)$

$\Rightarrow (2z-3y)^2 = 4a(z-3)(z-3x)^2$

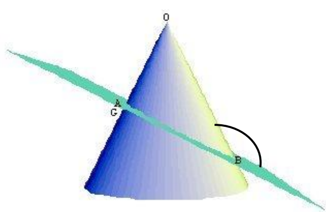
$\therefore\ \ (2z-3y)^2 = 4a(z-3)(z-3x)^2$ is the equation of the cone.

Model-3

Problem:7. Find the angle between the lines of section plane

$$x-3y+z=0\ and\ the\ cone\ \ x^2-5y^2+z^2=0$$

Solution:



Suppose the equation of line section of a cone $x^2-5y^2+z^2=0$ by the plane $x-3y+z=0$ is $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ where l, m, n are d cs of the line section.

$\therefore\ \ \ l-3m+n=0$ -----------(1) and $l^2-5m^2+n^2=0$ --------(2)

From (1) $l = 3m-n$ and to eliminate l in (2)

$(3m-n)^2-5m^2+n^2=0\ \Rightarrow\ \ 4m^2-6mn+2n^2=0\ \Rightarrow 2m^2-3mn+n^2=0$

$\Rightarrow (m-n)(2m-n)=0\ \Rightarrow\ \ m-n\ \ =0\ ---\ \ (3)$

$2m-n=0$ -----------(4)

To sole (1) & (3) $\quad l-3m+n\ \ =0 \qquad$ -3 1 1 -3

$m-n\ \ =0 \qquad$ 1 -1 0 1

$$\frac{l}{3-1} = \frac{m}{0+1} = \frac{n}{1+0} \Rightarrow \frac{l}{2} = \frac{m}{1} = \frac{n}{1}$$

To solve (1) & (4) $\quad l - 3m + n = 0 \qquad$ -3 1 1 -3

$\qquad 2m - n \ = 0 \qquad$ 2 -1 0 2

$$\frac{l}{3-2} = \frac{m}{0+1} = \frac{n}{2-0} \Rightarrow \frac{l}{1} = \frac{m}{1} = \frac{n}{2}$$

Hence the equations of line of intersections plane and a cone through origin is

$$\frac{x}{2} = \frac{y}{1} = \frac{z}{1} \ \& \ \frac{x}{1} = \frac{y}{1} = \frac{z}{2}$$

Direction cosines of the lines $\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}$ *and* $\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}$

Angle between the lines of section

$$Cos\,\theta = l_1 l_2 + m_1 m_2 + n_1 n_2$$

$$= \frac{2}{\sqrt{6}\sqrt{6}} + \frac{1}{\sqrt{6}\sqrt{6}} + \frac{2}{\sqrt{6}\sqrt{6}} = \frac{5}{6}$$

$$\Rightarrow \theta = \cos^{-1}\frac{5}{6}$$

Problem:8. Find the angle between the lines of section plane

$$2x + y - z = 0 \ and\ the\ cone\ 4x^2 - y^2 + 3z^2 = 0$$

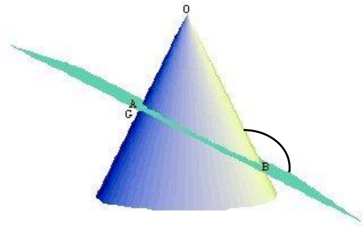
Solution: Suppose the equation of line section of a cone $4x^2 - y^2 + 3z^2 = 0$ by the plane

$2x + y - z = 0$ is $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ where l, m, n are d cs of the line section.

$\therefore \quad 2l + m - n = 0$ -----------(1) and $4l^2 - m^2 + 3n^2 = 0$ --------(2)

From (1) $n = 2l + m$ and to eliminate n in (2)

$4l^2 - m^2 + 3(2l + m)^2 = 0 \Rightarrow \ 8l^2 + 6lm + m^2 = 0$



$\Rightarrow (4l+m)(2l+m)=0 \;\Rightarrow (4l+m) \;=0 ---(3)$

$(2l+m)=0$ -----------(4)

To sole (1) & (3) $\quad 2l+m-n=0 \qquad$ 1 -1 2 1

$4l+m \;=0 \qquad$ 1 0 4 1

$$\frac{l}{0+1}=\frac{m}{-4+0}=\frac{n}{2-4}\Rightarrow \frac{l}{1}=\frac{m}{-4}=\frac{n}{-2}$$

To solve (1) & (4) $\quad 2l+m-n=0 \qquad$ 1 -1 2 1

$2l+m \;=0 \qquad$ 1 0 2 1

$$\frac{l}{0+1}=\frac{m}{-2+0}=\frac{n}{2-2}\Rightarrow \frac{l}{1}=\frac{m}{-2}=\frac{n}{0}$$

Hence the equations of line of intersections plane and a cone through origin is

$$\frac{x}{1}=\frac{y}{-4}=\frac{z}{-2} \;\&\; \frac{x}{1}=\frac{y}{-2}=\frac{z}{0}$$

Direction cosines of the lines $\frac{1}{\sqrt{21}},\frac{-4}{\sqrt{21}},\frac{-2}{\sqrt{21}}$ and $\frac{1}{\sqrt{5}},\frac{-2}{\sqrt{5}},\frac{0}{\sqrt{5}}$

Angle between the lines of section

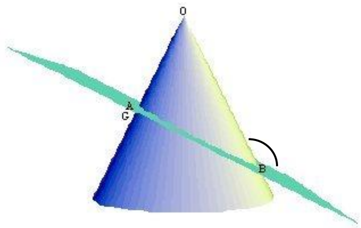
$$Cos\,\theta=l_1l_2+m_1\,m_2+n_1\,n_2$$

$$=\frac{1}{\sqrt{5}\sqrt{21}}+\frac{8}{\sqrt{5}\sqrt{21}}+\frac{0}{\sqrt{5}\sqrt{21}}=\frac{9}{\sqrt{105}}=\sqrt{\frac{81}{105}}=\sqrt{\frac{27}{35}}$$

$$\Rightarrow \theta=\cos^{-1}\sqrt{\frac{27}{35}}$$

Problem-9 Prove that if the angle between the lines of intersection of the plane $x+y+z=0$ and the cone $ayz+bzx+cxy=0$ is $\frac{\pi}{2}$ then $a+b+c=0$.

Solution:



Let the line of intersection of the plane and the cone is

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$$ where l, m, n are d cs of the line section.

$$\therefore \quad l + m + n = 0 \quad \text{---------------(1)}$$

$$amn + bnl + clm = 0 \quad \text{--------------(2)}$$

From (1) $n = -l - m$ and to eliminate in (2)

$$am(-l - m) + b(-l - m)l + clm = 0$$

$$-alm - am^2 - bl^2 - blm + clm = 0$$

$$-bl^2 - (a + b - c)lm - am^2 = 0$$

$$bl^2 + (a + b - c)lm + am^2 = 0$$

$b\,\frac{l^2}{m^2} + (a + b - c)\frac{l}{m} + a = 0$ is a quadratic equation in $\frac{l}{m}$

Suppose $\frac{l_1}{m_1}, \frac{l_2}{m_2}$ are the roots

Product of roots $\frac{l_1}{m_1} \cdot \frac{l_2}{m_2} = \frac{a}{b} \Rightarrow \frac{l_1 l_2}{m_1 m_2} = \frac{a}{b} \Rightarrow \frac{l_1 l_2}{a} = \frac{m_1 m_2}{b}$

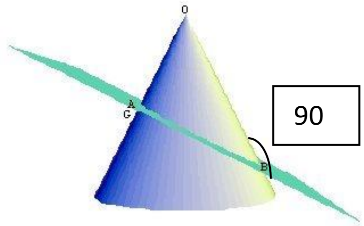
Similarly, $\frac{m_1 m_2}{b} = \frac{n_1 n_2}{c}$ and hence $\frac{l_1 l_2}{a} = \frac{m_1 m_2}{b} = \frac{n_1 n_2}{c} = k$

Since the lines are perpendicular, we get $l_1 l_2 + m_1 m_2 + n_1 n_2 = 0 \Rightarrow ka + kb + kc = 0$

$$\therefore\ a + b + c = 0$$

Problem-10 Prove that if the angle between the lines of intersection of the plane $ax + by + cz = 0$ and the cone $yz + zx + xy = 0$ is $\frac{\pi}{2}$ then $\frac{1}{a}. + \frac{1}{b} + \frac{1}{c} = 0$

Solution:



Let the line of intersection of the plane and the cone is

$\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ where l, m, n are d cs of the line section.

$\therefore \quad al + bm + cn = 0$ ---------------(1)

$mn + nl + lm = 0$ --------------(2)

From (1) $cn = -al - bm \quad \Rightarrow n = \frac{-al-bm}{c}$ and to eliminate in (2)

$$m(\frac{-al-bm}{c}) + (\frac{-al-bm}{c})l + lm = 0$$

$$-alm - bm^2 - al^2 - blm + clm = 0$$

$$-al^2 - (a + b - c)lm - bm^2 = 0$$

$$al^2 + (a + b - c)lm + bm^2 = 0$$

$a\frac{l^2}{m^2} + (a + b - c)\frac{l}{m} + b = 0$ is a quadratic equation in $\frac{l}{m}$

Suppose $\frac{l_1}{m_1}, \frac{l_2}{m_2}$ are the roots

Product of roots $\frac{l_1}{m_1} . \frac{l_2}{m_2} = \frac{b}{a} \Rightarrow \frac{l_1 l_2}{m_1 m_2} = \frac{b}{a} \Rightarrow \frac{l_1 l_2}{1/a} = \frac{m_1 m_2}{1/b}$

Similarly, $\frac{m_1 m_2}{1/b} = \frac{n_1 n_2}{1/c}$ and hence $\frac{l_1 l_2}{1/a} = \frac{m_1 m_2}{1/b} = \frac{n_1 n_2}{1/c} = k$

Since the lines are perpendicular, we get $l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$

$$\Rightarrow k(\frac{1}{a}) + k(\frac{1}{b}) + k\left(\frac{1}{c}\right) = 0 \Rightarrow \frac{1}{a}. + \frac{1}{b} + \frac{1}{c} = 0$$

Model-4

Enveloping Cone: Let S is a surface and P is a point not on the surface. The set of tangent lines to the surface S and passing through the point P form a cone with vertex P and is called Enveloping cone.

Theorem: Prove that the equation of Enveloping cone of the sphere

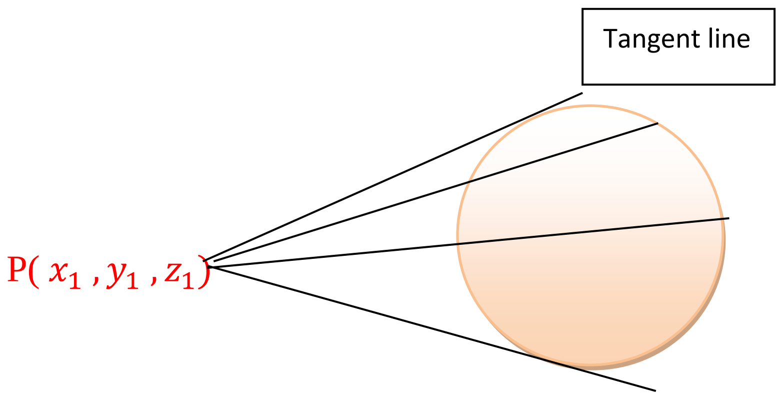
$x^2+y^2+z^2=a^2$ with vertex at (x_1, y_1, z_1) is

$(xx_1+y\,y_1+zz_1-a^2)^2=(x^2+y^2+z^2-a^2)({x_1}^2+{y_1}^2+{z_1}^2-a^2)$

Proof: Let S = $x^2+y^2+z^2-a^2$

$$S_1 = xx_1+y\,y_1+zz_1-a^2$$

$$S_{11}={x_1}^2+{y_1}^2+{z_1}^2-a^2$$



Equation of a line through the point P = (x_1, y_1, z_1) with drs l, m, n is

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}=r \text{ ----------(1)}$$

$\Rightarrow A(lr+x_1, mr+y_1, nr+z_1)$ *is any point on the line.*

If the line touch at this point, then A lies on the sphere $x^2+y^2+z^2-a^2$

$$\therefore (lr+x_1)^2+(mr+y_1)^2+(nr+z_1)^2-a^2=0$$

$r^2(l^2+m^2+n^2)+2r(l\,x_1+m\,y_1+n\,z_1)+({x_1}^2+{y_1}^2+{z_1}^2-a^2)=0$

Is quadratic equation in r . If the line is tangent to the sphere it touches the sphere at unique point. That is $b^2 - 4ac = 0$

$\Rightarrow\ 4(l\,x_1 + m\,y_1 + n\,z_1)^2 = 4(l^2 + m^2 + n^2)(x_1{}^2 + y_1{}^2 + z_1{}^2 - a^2)$

Using (1) to eliminate l, m, n

$\Rightarrow\ [(x - x_1)\,x_1 + (y - y_1)y_1 + (z - z_1)\,z_1]^2$

$= [(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2](x_1{}^2 + y_1{}^2 + z_1{}^2 - a^2)$

$\Rightarrow\ [\,xx_1 + y\,y_1 + zz_1 - (x_1{}^2 + y_1{}^2 + z_1{}^2)]^2$

$= [\,x^2 + y^2 + z^2 - 2(\,xx_1 + y\,y_1 + zz_1) + (x_1{}^2 + y_1{}^2 + z_1{}^2)]S_{11}$

$\Rightarrow\ [\,(S_1 + a^2) - (S_{11} + a^2)]^2$

$= [\,S + a^2 - 2(\,S_1 + a^2) + S_{11} + a^2]S_{11}$

$\Rightarrow\ [S_1 - S_{11}]^2 = [\,S - 2S_1 + S_{11}]\,S_{11}$

$\Rightarrow\ S_1{}^2 - 2S_1 S_{11} + S_{11}{}^2 = SS_{11} - 2S_1 S_{11} + S_{11}{}^2$

$\Rightarrow S_1{}^2 = SS_{11}$

Problem 11: Find the equation of Enveloping cone of the sphere $x^2 + y^2 + z^2 + 2x - 2y - 2 = 0$ whose vertex at (1, 1, 1) .

Solution: Given that vertex = $(\,x_1, y_1, z_1) = (1, 1, 1)$

Equation of the sphere S = $x^2 + y^2 + z^2 + 2x - 2y - 2$

$S_1 = xx_1 + y\,y_1 + zz_1 + (x + x_1) - (y + y_1) - 2$

$= x(1) + y\,(1) + z(1) + (x + 1) - (y + 1) - 2$

$= 2x + z - 2$

$S_{11} = x_1{}^2 + y_1{}^2 + z_1{}^2 + 2x_1 - 2y_1 - 2$

$= (1)^2 + (1)^2 + (1)^2 + 2(1) - 2(1) - 2 = 1$

Equation of Enveloping cone is $S_1{}^2 = SS_{11}$

$\Rightarrow\ (2x + z - 2)^2 = (x^2 + y^2 + z^2 + 2x - 2y - 2)(1)$

$\Rightarrow 3x^2 - y^2 + 4zx - 10x + 2y - 4z + 6 = 0.$

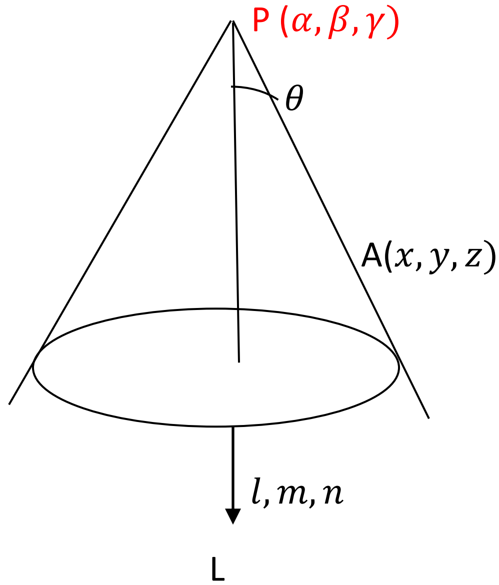
Model-5

Right circular cone : A right circular cone is a surface generated by a line which passes through a fixed point and makes a constant angle with a fixed line through the fixed point. The line L is called the axis, and the angle is called semi-vehicle angle of the cone

Theorem: Prove that the equation of right circular cone with vertex at (α, β, γ), the semi-vertical angle θ $and\ axie\ having\ direction\ ratios\ (l, m, n) is$

$$[l(x-\alpha)+m(y-\beta)+n(z-\gamma)]^2 = (l^2+\ m^2\ +\ \ n^2\ )[(x-\alpha)^2+\ (y-\beta)^2\ +\ \ (z-\gamma)^2\ \ ]cos^2\theta$$

Proof: In the diagram



Let P=(α, β, γ) is the vertex and A(x, y, z) any point on the cone

And drs of the axis are l, m, n also the semi-vertical angle **APL =** θ

Drs of the line AP = $x-\alpha, y-\beta, z-\gamma$

Drs of the axis AL = l, m, n

Angle between AL and AP is

$$\cos\theta = \frac{l(x-\alpha)+m(y-\beta)+n(z-\gamma)}{\sqrt{l^2+\ m^2\ +\ \ n^2}\ \sqrt{[(x-\alpha)^2+\ (y-\beta)^2\ +\ \ (z-\gamma)^2}}$$

Squaring on both sides and cross multiply we get

$$(l^2+\ m^2\ +\ \ n^2\ )[(x-\alpha)^2+\ (y-\beta)^2\ +\ \ (z-\gamma)^2\ \ ]cos^2\theta$$

$= [l(x-\alpha)+m(y-\beta)+n(z-\gamma)]^2$

Which is the required equation of right circular cone.

Problems 12. Find the equation of right circular cone whose vertex at P=(2,-3,5)axis which makes equal angles with the coordinate axes and which passes through A=(1,-2,3).

Solution: In the given problem Vertex P=$(\alpha,\beta,\gamma) = (2,-3,5)$

Drs of the axis = $cos\theta, cos\theta, cos\theta$ $\Rightarrow$ Drs of the axis=1,1,1

Suppose the semi-vertical angle = α

equation of right circular cone is

$$[l(x-\alpha)+m(y-\beta)+n(z-\gamma)]^2$$

$$= (l^2 + m^2 + n^2)[(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2]cos^2\theta$$

$$\Rightarrow [1(x-2)+1(y+3)+1(z-5)]^2$$

$$= (1^2 + 1^2 + 1^2)[(x-2)^2 + (y+3)^2 + (z-5)^2]cos^2\alpha$$

$$\Rightarrow [x+y+z-4]^2 = 3[(x-2)^2 + (y+3)^2 + (z-5)^2]cos^2\alpha \quad \text{---------}(1)$$

But it is posing through A=(1,-2,3).

$$[1-2+3-4]^2 = 3[(1-2)^2 + (-2+3)^2 + (3-5)^2]cos^2\alpha$$

$$4 = 3[1 + 1 + 4]cos^2\alpha \Rightarrow cos^2\alpha = \frac{2}{9} \Rightarrow cos\,\alpha = \frac{\sqrt{2}}{3}$$

Equation of Right circular cone (1) $\Rightarrow$

$$[x+y+z-4]^2 = 3[(x-2)^2 + (y+3)^2 + (z-5)^2](\frac{2}{9})$$

$$\Rightarrow x^2+y^2+z^2+6(xy+yz+zx)-16x-36y-4z-28=0$$

Problem 13. Find the equation of right circular cone whose vertex at P=(2,-3,5) axis which makes equal angles with the coordinate axes and the semi-vertical angle 30^0

Solution: In the given problem Vertex P=$(\alpha,\beta,\gamma) = (2,-3,5)$

Drs of the axis = $cos\theta, cos\theta, cos\theta$ $\Rightarrow$ Drs of the axis=1,1,1

Suppose the semi-vertical angle $\theta = 30^0$

equation of right circular cone is

$[l(x-\alpha)+m(y-\beta)+n(z-\gamma)]^2$

$= (l^2 + m^2 + n^2)[(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2]cos^2\theta$

$\Rightarrow [1(x-2)+1(y+3)+1(z-5)]^2$

$= (1^2 + 1^2 + 1^2)[(x-2)^2 + (y+3)^2 + (z-5)^2]cos^2 30^0$

$\Rightarrow [x+y+z-4]^2 = 3[(x-2)^2 + (y+3)^2 + (z-5)^2](\frac{3}{4})$---------(1)

Equation of Right circular cone (1) ⇒

$[x+y+z-4]^2 = 3[x^2-4x+4+y^2+6y+9+z^2-10z+25](\frac{3}{4})$

$\Rightarrow (t-4)^2 = \frac{9}{4}(x^2+y^2+z^2-4x+6y-10z+38)$

$\Rightarrow 4(t^2-8t+16) = 9(x^2+y^2+z^2-4x+6y-10z+38)$ but $t = x+y+z$

$\Rightarrow 4[x^2+y^2+z^2+2xy+2yz+2zx-8(x+y+z)+16]$

$= 9(x^2+y^2+z^2-4x+6y-10z+38)$

$\Rightarrow 5(x^2+y^2+z^2)-8(xy+yz+zx)-4x+86y-58z+278=0$

Model-6

Problem 14: Finding the vertex of the cone

$7x^2 + 2y^2 + 2z^2 - 10zx + 10xy + 26x - 2y + 2z - 17 = 0.$

Solution: The given equation converts into homogeneous equation

$7x^2 + 2y^2 + 2z^2 - 10zx + 10xy + 26xt - 2yt + 2zt - 17t^2 = 0$. Where t =1

Let $f = 7x^2 + 2y^2 + 2z^2 - 10zx + 10xy + 26xt - 2yt + 2zt - 17t^2 = 0$

$\frac{\partial f}{\partial x} = 14x - 10z + 10y + 26t = 14x + 10y - 10z + 26 = 0 - - - (1)$

$\frac{\partial f}{\partial y} = 4y + 10x - 2t = 10x + 4y - 2 = 0 - - - - (2)$

$\frac{\partial f}{\partial z} = 4z - 10x + 2t = -10x + 4z + 2 = 0$ ----------(3)

To solve the equations (1) and (2)

$2\times(1) \Rightarrow \quad 28x + 20y - 20z + 52 = 0$

$5\times(2) \Rightarrow \quad 50x + 20y \qquad - 10 = 0$

Subtracting $-22x - 20z + 62 = 0$

$5 \times (3) \Rightarrow -50x + 20z + 10 = 0$ subtracting them $-72x + 72 = 0 \Rightarrow x = 1$

From the equation (3) $-10(1) + 4z + 2 = 0 \Rightarrow 4z = 8 \Rightarrow z = 2$

$$10x + 4y - 2 = 0 \Rightarrow 10(1) + 4y - 2 = 0 \Rightarrow 4y = -8 \Rightarrow\Rightarrow y = -2$$

$$\therefore vertex\ of\ the\ cone = (1, -2, 2)$$

Problem 15.Finding the vertex of the cone

$4x^2 - y^2 + 2z^2 + 2xy - 3yz + 12x - 11y + 6z + 4 = 0.$

Solution: The given equation converts into homogeneous equation

$4x^2 - y^2 + 2z^2 + 2xy - 3yz + 12xt - 11yt + 6zt + 4t^2 = 0.$ Where t =1

Let $f = 4x^2 - y^2 + 2z^2 + 2xy - 3yz + 12xt - 11yt + 6zt + 4t^2 = 0.$

$\frac{\partial f}{\partial x}$ $=8x + 2y + 12t = 4x + y + 6 = 0 - - - (1)$

$\frac{\partial f}{\partial y} = -2y + 2x - 3z - 11t = 2x - 2y - 3z - 11 = 0 \ - - - -(2)$

$\frac{\partial f}{\partial z} = 4z - 3y + 6t = -3y + 4z + 6 = 0$ ----------(3)

To solve the equations (1) and (2)

$(1) \Rightarrow \ 4x + y + 6 = 0$

$2 \times \ (2) \Rightarrow \quad 4x - 4y - 6z - 22 = 0$

Subtracting them $5y + 6z + 28 = 0$ ----------(4)

$3 \times (4) \Rightarrow 15y + 18z + 84 = 0$

$5 \times (3) \Rightarrow \ -15y + 20z + 30 = 0$

add them $38z + 114 = 0 \Rightarrow z = -3$

from (4) $5y + 6(-3) + 28 = 0 \Rightarrow 5y + 10 = 0 \Rightarrow y = -2$

$$from(1)\ 4x + y + 6 = 0 \ \Rightarrow 4x + (-2) + 6 = 0 \Rightarrow x = -1$$

$$\therefore vertex\ of\ the\ cone = (-1, -2, -3\ )$$

Problem 16: Finding the vertex of the cone

$2y^2 - 8yz - 4zx - 8xy + 6x - 4y - 2z + 5 = 0.$

Solution: The given equation converts into homogeneous equation

$2y^2 - 8yz - 4zx - 8xy + 6xt - 4yt - 2zt + 5t^2 = 0.$ where t =1

Let $f = 2y^2 - 8yz - 4zx - 8xy + 6xt - 4yt - 2zt + 5t^2 = 0$

$\frac{\partial f}{\partial x} = -4z - 8y + 6t = 4y + 2z - 3 = 0 - - - (1)$

$\frac{\partial f}{\partial y} = 4y - 8z - 8x - 4t = 0 = 2x - y + 2z + 1 = 0 \ - - - - (2)$

$\frac{\partial f}{\partial z} = -8y - 4x - 2t = 0 \Rightarrow 2x + 4y + 1 = 0$ ----------(3)

To solve the equations (2) and (3)

$(2) - (3) \Rightarrow \ -5y + 2z = 0 \ - - - - (4)$

$(1) - (4) \Rightarrow \quad 9y - 3 = 0 \Rightarrow y = \frac{1}{3}$

From (4) -5(1/3) + 2z= 0 $\Rightarrow z = \frac{5}{6}$

From (3) $2x + 4\left(\frac{1}{3}\right) + 1 = 0 \Rightarrow 2x + \frac{7}{3} = 0 \Rightarrow x = -\frac{7}{6}$

Vertex of the cone = $\left(\frac{-7}{6}, \frac{1}{3}, \frac{5}{6}\right)$

Model:7

Reciprocal Cones

Equation of tangent plane to the cone S = 0 at the point P = (x_1, y_1, z_1) is $S_1 = 0$

Definition **(Reciprocal cone):**

The locus of lines through the vertex of the cone

$ax^2 + by^2 + cz^2 + 2hxy + 2fyz + 2gzx = 0$ perpendicular to its tangent planes is the cone called Reciprocal cone.

The equation of reciprocal cone of the cone

$$ax^2 + by^2 + cz^2 + 2hxy + 2fyz + 2gzx = 0 \; is$$

$$Ax^2 + By^2 + Cz^2 + 2Hxy + 2Fyz + 2Gzx = 0$$

where A,B,C,H,G,F are the co-factors of a, b, c, h, g, f in the Matrix

$$\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$$

that is $A = bc - f^2$ $\quad B = ac - g^2$ $\quad C = ab - h^2$

H = $-(hc - gf)$ $\quad G = hf - bg,$ $\quad F = -(af - hg)$

Problem 17: Find the equation of the reciprocal cone of the cone

$$4x^2 - y^2 + 2z^2 + 2xy - 3yz + 6zx = 0$$

Solution: Given that $4x^2 - y^2 + 2z^2 + 2xy - 3yz + 6zx = 0$

Let $a = 4 \quad b = -1 \quad c = 2 \quad 2h = 2, \; 2f = -3 \quad 2g = 6$

$a = 4 \quad b = -1 \quad c = 2 \quad h = 1, \; f = -3/2 \quad g = 3$

The equation of reciprocal cone of the cone $ax^2 + by^2 + cz^2 + 2hxy + 2fyz + 2gzx = 0$ *is* $Ax^2 + By^2 + Cz^2 + 2Hxy + 2Fyz + 2Gzx = 0$ *where A, B, C, H, G, F are the co-factors of a, b, c, h, g, f in the Matrix*

$$\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$$

that is $A = bc - f^2 = -2 - \frac{9}{4} = -\frac{17}{4}$

$B = ac - g^2 = 8 - 9 = -1$

$C = ab - h^2 = -4 - 1 = -5$

H = $-(hc - gf) = -(2 — \frac{9}{2}) = \frac{5}{2}$

$G = hf - bg = -\frac{3}{2} + 3 = \frac{3}{2}$

$F = -(af - hg) = -(-\frac{12}{2} - 3) = 9$

$Ax^2 + By^2 + Cz^2 + 2Hxy + 2Fyz + 2Gzx=0$

$-\frac{17}{4}x^2 - 1y^2 - 5z^2 + 2(\frac{5}{2})xy + 2(9)yz + 2(\frac{3}{2})zx=0$

$-17x^2 - 4y^2 - 20z^2 + 20xy + 72yz + 12zx=0$

$17x^2 + 4y^2 + 20z^2 - 20xy - 72yz - 12zx=0$

Problem 18. Show that the general equation of a cone which touches the three coordinate axes is

$$\sqrt{\mathrm{fx}} \pm \sqrt{\mathrm{gy}} \pm \sqrt{\mathrm{hz}}, \qquad \mathrm{f, g, h} \text{ being parameters.}$$

Solution: We that the equation of a cone whose generators are coordinate axes is

$$hxy + fyz + gzx = 0$$

$$i.e.\ 0x^2 + 0y^2 + 0z^2 + 2hxy + 2fyz + 2gzx = 0$$

$$a = 0 \quad b = 0 \quad c = 0 \quad h = h \quad f = f \quad g = g$$

Now equation of reciprocal cone

$Ax^2 + By^2 + Cz^2 + 2Hxy + 2Fyz + 2Gzx=0$ *where A,B,C,H,G,F are the co-factors of a,b,c,h,g,f in the Matrix*

$$\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$$

that is $A = bc - f^2 = 0 - f^2 = -f^2$

$B = ac - g^2 = 0 - g^2 = -g^2$

$C = ab - h^2 = 0 - h^2 = -h^2$

$\mathrm{H} = -(hc - gf) = 0 + gf = gf$

$G = hf - bg = hf - 0 = hf$

$F = -(af - hg) = 0 + hg = hg$

$\therefore$ *Equation of reciprocal cone*

$$-f^2x^2 - g^2y^2 - h^2z^2 + 2gfxy + 2hgyz + 2hfzx = 0$$

$$\Rightarrow f^2x^2 + g^2y^2 + h^2z^2 - 2gfxy - 2hgyz - 2hfzx = 0$$

$$\Rightarrow f^2x^2 + g^2y^2 + h^2z^2 - 2gfxy - 2hgyz - 2hfzx + 2hfzx - 2hfzx = 0$$

$$\Rightarrow (fx - gy + hz)^2 = 4hfzx$$

$$\Rightarrow fx - gy + hz = \pm 2\sqrt{fx}\,\sqrt{hz}$$

$$\Rightarrow fx \pm 2\sqrt{fx}\,\sqrt{hz} + hz = gy$$

$$\Rightarrow (\sqrt{fx} \pm \sqrt{hz})^2 = [\sqrt{gy}]^2$$

$$\Rightarrow \sqrt{fx} \pm \sqrt{hz} = \pm\sqrt{gy}$$

$$\Rightarrow \sqrt{fx} \pm \sqrt{gy} \pm \sqrt{hz} = 0$$

Note: Condition for a cone $ax^2 + by^2 + cz^2 + 2hxy + 2fyz + 2gzx = 0$ having three mutually perpendicular generators is $a + b + c = 0$.

Problem 19: Show that if a right circular cone has set of three mutually perpendicular generators its semi vertical angle must be $\tan^{-1}\sqrt{2}$

Solution: *Suppose* (0,0,0) *is vertex and* l, m, n *are drs and* θ *is the semiverical angle of a right circular cone then its equation*

$$[l(x-\alpha) + m(y-\beta) + n(z-\gamma)]^2$$

$$= (l^2 + m^2 + n^2)[(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2]cos^2\theta$$

$$[lx + my + nz]^2 = (l^2 + m^2 + n^2)[x^2 + y^2 + z^2]cos^2\theta$$

But it having three mutually perpendicular generators

$$\therefore coeff\ of\ x^2 + coeff\ of\ y^2 + coeff\ of\ z^2 = 0$$

$$\Rightarrow l^2 - (l^2 + m^2 + n^2)cos^2\theta$$

$$+ m^2 - (l^2 + m^2 + n^2)cos^2\theta$$

$$+ n^2 - (l^2 + m^2 + n^2)cos^2\theta = 0$$

$$\Rightarrow (l^2 + m^2 + n^2) - 3(l^2 + m^2 + n^2)cos^2\theta = 0$$

$$(l^2 + m^2 + n^2)[1 - 3cos^2\theta]) = 0$$

Since $(l^2 + m^2 + n^2) \neq 0 \quad \therefore [1 - 3cos^2\theta]) = 0$

$\Rightarrow cos^2\theta = \frac{1}{3} \Rightarrow sec^2\theta = 3 \Rightarrow sec\theta = \sqrt{3}$

$\Rightarrow tan^2\theta = sec^2\theta - 1 = 3 - 1 = 2$

$\Rightarrow \tan\theta = \sqrt{2} \Rightarrow \theta = \tan^{-1}\sqrt{2}$

All the Best