GOVERNMENT DEGREE COLLEGE, RAVULAPALEM
NAAC Accredited with 'B' Grade(2.61 CGPA)
(Affiliated to Adikavi Nannaya University)
Beside NH-16, Main Road, Ravulapalem-533238, East Godavari Dist., A.P, INDIA
E-Mail : jkcjyec.ravulapalem@gmail.com, Phone : 08855-257061
ISO 50001:2011, ISO 14001:2015, ISO 9001:2015 Certified College

DIFFERENTIAL EQUATIONS

B.SRINIVASARAO.GDC RVPM

UNIT-1. DIFFERENTIAL EQUATIONS OF FIRST ORDER AND FIRST DEGREE

Definition:

A differential equation is in the form $\frac{dy}{dx} = f(x, y)$ called a differential equation of the first order and the first degree. There are four types

1.Linear differential equations.

2.Bernoullis Differential equations.

3.Exact Differential Equations.

4.Non- Exact Differential Equations.

1.Linear differential equations:

A differential equation is in the form $\frac{dy}{dx} + P(x)\, y = Q(x)$ where P and Q are functions of x , is called Linear differential equation.

Working Rule to solve the equation:

1.First to reduce the given differential equation in the form $\frac{dy}{dx} + P\, y = Q$

and then identify P and Q.

2.To find the Integrating Factor (I.F) = $e^{\int P\, dx}$

3.And then to find the general solution by $y(I.F) = \int Q(I.F)dx + c.$

Problems:

1.Solve $\frac{dy}{dx} + 2xy = e^{-x^2}$

Solution: Given differential equation $\frac{dy}{dx} + 2xy = e^{-x^2}$

Let P = $2x$ & Q = e^{-x^2}

The integrating factor (I.F) = $e^{\int P\,dx} = e^{\int 2x\,dx} = e^{2\left(\frac{x^2}{2}\right)} = e^{x^2}$

I F = e^{x^2}

The general solution is $y(I.F) = \int Q(I.F)dx + c.$

$$y\left(e^{x^2}\right) = \int e^{-x^2}\left(e^{x^2}\right)dx + c = \int 1dx + c = x + c$$

$$\therefore ye^{x^2} = x + c$$

2.Solve $x\log x\,\frac{dy}{dx} + y = 2\log x$

Solution: Given differential equation $x\log x\,\frac{dy}{dx} + y = 2\log x$

It can be reduced to $\frac{dy}{dx} + \frac{1}{x\log x}\,y = \frac{2}{x}$

Let P = $\frac{1}{x\log x}$ & Q = $\frac{2}{x}$

The integrating factor (I.F) = $e^{\int P\,dx} = e^{\int \frac{1}{x\log x}\,dx} = e^{\log(\log x)} = \log x$

I F = $\log x$

The general solution is $y(I.F) = \int Q(I.F)dx.$ $\quad let\ log\ x = t \Rightarrow \frac{1}{x}\,dx = dt$

$$y\log x = \int \frac{2}{x}\,(\log x)dx = 2\int t\,dt = 2\left(\frac{t^2}{2}\right) + c = t^2 + c = (\log x)^2 + c$$

$\therefore$ The general solution $y \log x = (\log x)^2 + c$

3.Solve $x\frac{dy}{dx} + 2y = x^2 \log x$

Solution: Given differential equation $x\frac{dy}{dx} + 2y = x^2 \log x$

It can be reduced to $\frac{dy}{dx} + \frac{2}{x}\, y = x \log x$

Let P = $\frac{2}{x}$ & Q = $x \log x$

The integrating factor (I.F) = $e^{\int P\, dx} = e^{\int \frac{2}{x}\, dx} = e^{2\log x} = e^{\log x^2} = x^2$

I F = x^2

The general solution is $y(I.F) = \int Q(I.F)dx + c.$

$$yx^2 = \int x \log x\ (x^2)dx = \int x^3 \log x\ dx + c$$

$$= \int \log x\ (x^3)\ dx + c$$

$$= (\,log\ x)\frac{x^4}{4} - \int \frac{1}{x}\left(\frac{x^4}{4}\right)dx + c$$

$$= \frac{x^4}{4}(\,log\ x) - \frac{1}{4}\int x^3\ dx + c$$

$$yx^2 = \frac{x^4}{4}(\,log\ x) - \frac{1}{4}\ \frac{x^4}{4} + c$$

$$= \frac{x^4}{4}(\,log\ x) - \frac{x^4}{16} + c$$

$\therefore$ The general solution $yx^2 = \frac{x^4}{4}(\,log\ x) - \frac{x^4}{16} + c$

4.Solve $(\,x^2+1)\frac{dy}{dx} + 4xy = \frac{1}{x^2+1}$

Solution: Given differential equation $(\,x^2+1)\frac{dy}{dx} + 4xy = \frac{1}{x^2+1}$

It can be reduced to $\frac{dy}{dx} + \frac{4x}{(\,x^2+1)}\, y = \frac{1}{(x^2+1)^2}$

Let P = $\frac{4x}{(\,x^2+1)}$ & Q = $\frac{1}{(x^2+1)^2}$

The integrating factor (I.F) = $e^{\int P\, dx} = e^{\int \frac{4x}{(x^2+1)}dx} = e^{2\int \frac{2x}{(x^2+1)}dx}$

$$= e^{2\log(x^2+1)} = e^{\log(x^2+1)^2}$$

$$\text{I F} = (x^2+1)^2$$

The general solution is $y(I.F) = \int Q(I.F)dx + c$.

$$y(x^2+1)^2 = \int (x^2+1)^2 \frac{1}{(x^2+1)^2} dx + c = \int 1\, dt + c = x + c$$

∴ The general solution $y(x^2+1)^2 = x + c$

5.Solve $(x+1)\frac{dy}{dx} - xy = 1 - x$

Solution: Given differential equation $(x+1)\frac{dy}{dx} - xy = 1 - x$

It can be reduced to $\frac{dy}{dx} - \frac{x}{(1+x)} y = \frac{1-x}{1+x}$

Let P = $-\frac{x}{(1+x)}$ and Q = $\frac{1-x}{1+x}$

The integrating factor (I.F) = $e^{\int P\,dx} = e^{-\int \frac{x}{(1+x)}dx} = e^{-\int \frac{1+x-1}{(1+x)}dx}$

$$= e^{-\int 1 - \frac{1}{(1+x)} dx}$$

$$= e^{-[x - \log(1+x)]}$$

$$= e^{-[x - \log(1+x)]}$$

$$= e^{-x} e^{\log(1+x)} = (1+x)e^{-x}$$

$$\text{I F} = (1+x)e^{-x}$$

The general solution is $y(I.F) = \int Q(I.F)dx + c$.

$$y(1+x)e^{-x} = \int (1+x)e^{-x}\left[\frac{1-x}{1+x}\right] dx + c = \int (1-x)e^{-x}\, dt + c$$

$$= (1-x)(-e^{-x}) - \int (-1)(-e^{-x})dx + \text{c}$$

$$= -(1-x)e^{-x} - \int e^{-x}dx + c$$

$$y(1+x)e^{-x} = -(1-x)e^{-x} + e^{-x} + c = xe^{-x} + c$$

∴ The general solution $y(1+x)e^{-x} = xe^{-x} + c$

6.Solve $x\frac{dy}{dx} + ylog\, x = e^x \; x^{1-\frac{1}{2}\log x}$

Solution: Given differential equation $x\frac{dy}{dx} + ylog\, x = e^x \; x^{1-\frac{1}{2}\log x}$

It can be reduced to $\frac{dy}{dx} + y\;\frac{logx}{x} \;=\; \frac{e^x\, x^{1-\frac{1}{2}\log x}}{x} = e^x \;\; x^{-\,\frac{1}{2}\log x}$

Let P = $\frac{logx}{x}$ & Q = $e^x \;\; x^{-\,\frac{1}{2}\log x}$

The integrating factor (I.F) = $e^{\int P\, dx} = e^{\int \frac{logx}{x}\; dx}$

$$= e^{\frac{(\log x)^2}{2}}$$

$$= e^{\log x\, \frac{\log x}{2}} = \left[e^{\log x}\right]^{\frac{logx}{2}} = x^{\;\frac{1}{2}\,\log x}$$

$$\text{I F} = x^{\;\frac{1}{2}\,\log x}$$

The general solution is $y(I.F) = \int Q(I.F)dx + c.$

$$y\, x^{\;\frac{1}{2}\,\log x} = \int\, e^x \;\; x^{-\,\frac{1}{2}\log x}\left(x^{\;\frac{1}{2}\,\log x}\right) dx + \text{ c}$$

$$= \int e^x\, d\,x \;+ c = \;\; e^x + c$$

The general solution $y\, x^{\;\frac{1}{2}\,\log x} \;=\; e^x + c$

7. Solve $x^2\,\frac{dy}{dx} + (x-2)y = x^2\, e^{-\frac{2}{x}}$

Solution: Given differential equation $x^2\,\frac{dy}{dx} + (x-2)y = x^2\, e^{-\frac{2}{x}}$

It can be reduced to $\frac{dy}{dx} + \frac{1}{x^2}\,(x-2)y = \; e^{-\frac{2}{x}}$

Let P = $\frac{x-2}{x^2} \;=\; \frac{1}{x} - \frac{2}{x^2}$ & Q = $e^{-\frac{2}{x}}$

The integrating factor (I.F) = $e^{\int P\, dx} = e^{\int [\,\frac{1}{x} - \frac{2}{x^2}\,]dx} = e^{log\, x + \frac{2}{x}}$

$$= \; e^{\log x}\, e^{\frac{2}{x}} \;= x\, e^{\;\frac{2}{x}}$$

$$\text{I F} = x\, e^{\frac{2}{x}}$$

The general solution is $y(I.F) = \int Q(I.F)dx + c.$

$$y\ x\, e^{\frac{2}{x}} = \int x\, e^{\frac{2}{x}} e^{-\frac{2}{x}} dx + c = \int x\, dt + c = \frac{x^2}{2} + c$$

$\therefore$ The general solution $y\ x\, e^{\frac{2}{x}} = \frac{x^2}{2} + c$

8.. Solve $(1 + x^2)\frac{dy}{dx} + 2xy = 4x^2$

Solution: Given differential equation $(1 + x^2)\frac{dy}{dx} + 2xy = 4x^2$

It can be reduced to $\frac{dy}{dx} + \frac{2x}{(1+x^2)}\ y = \frac{4x^2}{1+x^2}$

Let P = $\frac{2x}{(1+x^2)}$ & Q = $\frac{4x^2}{1+x^2}$

The integrating factor (I.F) = $e^{\int P\,dx} = e^{\int \frac{2x}{(x^2+1)}dx} = e^{\log(1+x^2)} = 1 + x^2$

$$\text{I F} = 1 + x^2$$

The general solution is $y(I.F) = \int Q(I.F)dx + c.$

$$y(1 + x^2) = \int (1 + x^2)\left[\frac{4x^2}{1+x^2}\right] dx + c = \int 4x^2\, dt + c = \frac{4x^3}{3} + c$$

$\therefore$ The general solution $y(1 + x^2) = \frac{4x^3}{3} + c$

9.Solve $cos^2x\ \frac{dy}{dx} + y = \tan x$

Solution: Given differential equation $cos^2x\ \frac{dy}{dx} + y = \tan x$

It can be reduced to $\frac{dy}{dx} + \frac{1}{cos^2x}\ y = \frac{Tan\,x}{cos^2x} \Rightarrow \frac{dy}{dx} + sec^2x\ \ y = \tan x\ sec^2x$

Let P = sec^2x & Q = $\tan x\ sec^2x$

The integrating factor (I.F) = $e^{\int P\,dx} = e^{\int sec^2x\,dx} = e^{Tan\,x}$

$$\text{I F} = e^{Tan\,x}$$

The general solution is $y(I.F) = \int Q(I.F)dx + c.$

$$ye^{Tan\,x} = \int \tan x\, sec^2 x\ e^{Tan\,x} dx + c$$

Let $\tan x = t \ \Rightarrow\ sec^2 x\, dx = dt$

$\therefore\ ye^{Tan\,x} = \int t\, e^t\ dt\ + c = e^t(t-1) + c$

$\therefore$ The general solution $ye^{Tan\,x} = e^{Tan\,x}(\tan x - 1) + c$

10.Solve $(x + 2y^3)\frac{dy}{dx} = y.$

Solution: To convert the differential equation in the form $\frac{dx}{dy} + P\,x = Q$

Given that $(x + 2y^3)\frac{dy}{dx} = y\ \Rightarrow y\frac{dx}{dy} = x + 2y^3$

$$\Rightarrow y\frac{dx}{dy} - x = 2y^3$$

$$\Rightarrow \frac{dx}{dy} - \frac{x}{y} = 2y^2$$

$Let\ P\ = \frac{-1}{y}\quad and\ Q\ =\ 2y^2$

The integrating factor (I.F) = $e^{\int P\,dy} = e^{\int \frac{-1}{y} dy} = e^{-\log y} = \frac{1}{y}$ $\therefore$ I F $= \frac{1}{y}$

The general solution $x(I.F) = \int Q(I.F)dy + c$

$$x\left(\frac{1}{y}\right) = \int 2y^2\left(\frac{1}{y}\right)dy + c = \int 2y dy + c = \frac{2y^2}{2} + c = y^2 + c$$

Hence the general solution $\frac{x}{y} = y^2 + c$

11.Solve $(1 + y^2)dx = (\tan^{-1} y\ - x)dy.$

Solution: To convert the differential equation in the form $\frac{dx}{dy} + P\,x = Q$

Given that $(1 + y^2)dx = (\tan^{-1} y\ - x)dy$

$$\Rightarrow (1 + y^2)\frac{dx}{dy} = (\tan^{-1} y\ - x)$$

$$\Rightarrow (1 + y^2)\frac{dx}{dy} + x = \tan^{-1} y\ \Rightarrow\ \frac{dx}{dy} + \frac{1}{1+y^2}\,x = \frac{\tan^{-1} y}{1+y^2}$$

$Let\ P\ = \frac{1}{1+y^2}\quad and\ Q\ = \frac{\tan^{-1} y}{1+y^2}$

The integrating factor (I.F) = $e^{\int P\,dy} = e^{\int \frac{1}{1+y^2}dy} = e^{\tan^{-1}y}$ $\therefore$ I F = $e^{\tan^{-1}y}$

The general solution $x(I.F) = \int Q(I.F)dy + c$

$x\left(e^{\tan^{-1}y}\right) = \int \frac{\tan^{-1}y}{1+y^2}\left(e^{\tan^{-1}y}\right)dy + c$ ------(1)

Let $\tan^{-1}y = t \Rightarrow \frac{1}{1+y^2}dy = dt$

(1) Can be written as $x\left(e^{\tan^{-1}y}\right) = \int te^t dt + c = e^t(t-1) + c$

Hence general solution is $x\left(e^{\tan^{-1}y}\right) = e^{\tan^{-1}y}(\tan^{-1}y - 1) + c$

2.Bernoulli's Differential equations

A differential equation is in the form $\frac{dy}{dx} + P(x)\,y = Q(x)y^n$ where P and Q are functions of x ,is called Bernoulli's differential equation.

Working Rule to solve the equation:

1.First to divide the DE with y^n *on both sides* $\frac{1}{y^n}\frac{dy}{dx} + P\,y^{1-n} = Q$ -----(*)

2. Put $y^{1-n} = z$ and then $(1-n)y^{-n}\frac{dy}{dx} = \frac{dz}{dx} \Rightarrow \frac{1}{y^n}\frac{dy}{dx} = \frac{1}{1-n}\frac{dz}{dx}$

3.Substitute the values in $(*)$

$$\frac{1}{1-n}\frac{dz}{dx} + P(x)\,z = Q(x) \text{ is a linear differential equation and then solve.}$$

Problems:

1. Solve $\frac{dy}{dx} + \frac{y}{x} = x^2y^6$

Solution: The given differential equation $\frac{dy}{dx} + \frac{y}{x} = x^2y^6$

Divide the DE with y^6 *on both sides* $\frac{1}{y^6}\frac{dy}{dx} + \frac{y^{-5}}{x} = x^2$ -------(1)

Put $y^{-5} = z$ and then $-5y^{-5-1}\frac{dy}{dx} = \frac{dz}{dx} \Rightarrow \frac{1}{y^6}\frac{dy}{dx} = \frac{1}{-5}\frac{dz}{dx}$

(1) Can be written as $\frac{-1}{5}\frac{dz}{dx} + \frac{z}{x} = x^2 \Rightarrow \frac{dz}{dx} - \frac{5z}{x} = -5x^2$

Let $P_1 = \frac{-5}{x}$ $\quad$ $and \;\; Q_1 = -5x^2$

The integrating factor (I.F) = $e^{\int P_1 \, dx}$

$$= e^{\int \frac{-5}{x} dx} = e^{-5 \int \frac{1}{x} dx}$$

$$= e^{-5 \log x} = e^{\log x^{-5}} = x^{-5} = \frac{1}{x^5} \qquad I\,F = \frac{1}{x^5}$$

$The\ general\ solution: z(I.F) = \int Q_1(I.F)dx + c$

$$y^{-5} \; \frac{1}{x^5} = \int -5x^2 \left(\frac{1}{x^5}\right) dx + c$$

$$\frac{1}{x^5\,y^5} = -5 \int x^{-3} dx + c = -5\left[\frac{x^{-3+1}}{-3+1}\right] + c$$

The general solution $\frac{1}{x^5\,y^5} = \frac{5}{2x^2} + c$

2. Solve $\frac{dy}{dx} + \frac{y}{x} = y^2 x \sin x$

Solution: The given differential equation $\frac{dy}{dx} + \frac{y}{x} = y^2 x \sin x$

Divide the DE with $y^2 on\ both\ sides$ $\frac{1}{y^2}\frac{dy}{dx} + \frac{y^{-1}}{x} = x \sin x$ -------(1)

Put $y^{-1} = z$ and then $(-1)\, y^{-1-1} \frac{dy}{dx} = \frac{dz}{dx} \Rightarrow \frac{1}{y^2}\frac{dy}{dx} = -\frac{dz}{dx}$

(1) Can be written as - $\frac{dz}{dx} + \frac{z}{x} = x \sin x \Rightarrow \frac{dz}{dx} - \frac{z}{x} = -x \sin x$

Let $P_1 = \frac{-1}{x}$ $\quad$ $and \;\; Q_1 = -\,x \sin x$

The integrating factor (I.F) = $e^{\int P_1 \, dx}$

$$= e^{\int \frac{-1}{x} dx}$$

$$= e^{-\log x} = e^{\log x^{-1}} = x^{-1} = \frac{1}{x} \qquad I\,F = \frac{1}{x}$$

$The\ general\ solution: z(I.F) = \int Q_1(I.F)dx + c$

$$y^{-1} \; \frac{1}{x} = \int -x \sin x \left(\frac{1}{x}\right) dx + c$$

$$\frac{1}{xy} = -\int \sin x \, dx + c = \cos x + c$$

Hence the GS is $\frac{1}{xy} = \cos x + c$

3. Solve $x\,\frac{dy}{dx} + y = y^2 \log x$

Solution: The given differential equation $x\,\frac{dy}{dx} + y = y^2 \log x$

Divide the DE with xy^2 *on both sides* $\frac{1}{y^2}\frac{dy}{dx} + \frac{y^{-1}}{x} = \frac{\log x}{x}$ -------(1)

Put $y^{-1} = z$ and then $(-1)\,y^{-1-1}\frac{dy}{dx} = \frac{dz}{dx} \Rightarrow \frac{1}{y^2}\frac{dy}{dx} = -\frac{dz}{dx}$

(1) Can be written as $-\frac{dz}{dx} + \frac{z}{x} = \frac{\log x}{x} \Rightarrow \frac{dz}{dx} - \frac{z}{x} = -\frac{\log x}{x}$

Let $P_1 = \frac{-1}{x}$ and $Q_1 = -\frac{\log x}{x}$

The integrating factor (I.F) = $e^{\int P_1\,dx}$

$$= e^{\int \frac{-1}{x} dx} = e^{-\log x} = e^{\log x^{-1}} = x^{-1} = \frac{1}{x} \qquad \boldsymbol{IF = \frac{1}{x}}$$

The general solution: $z(I.F) = \int Q_1(I.F)dx + c$

$$y^{-1}\,\frac{1}{x} = \int -\frac{\log x}{x}\left(\frac{1}{x}\right)dx + c = -\int \frac{1}{x^2}\log x\;dx + c$$

$$\frac{1}{xy} = -\left[\log x\;\{-\frac{1}{x}\} - \int \frac{1}{x}\left\{-\frac{1}{x}\right\}dx\;\right] + c$$

$$\frac{1}{xy} = \frac{1}{x}\;\log x - \int \frac{1}{x^2}\,dx + c = \frac{1}{x}\;\log x + \frac{1}{x} + c$$

Hence the GS is $\frac{1}{xy} = \frac{1}{x}\,[\;\log x + 1] + c$

4. *solve* $3\frac{dy}{dx} + \frac{2y}{1+x} = \frac{x^3}{y^2}$

Solution: *Multiply the given differential equation with* $\frac{y^2}{3}$

$$y^2\frac{dy}{dx} + \frac{2y^3}{3(1+x)} = \frac{x^3}{3} \text{ ----------(1)}$$

Put $y^3 = z$ and then $3y^2\frac{dy}{dx} = \frac{dz}{dx} \Rightarrow y^2\,\frac{dy}{dx} = \frac{1}{3}\frac{dz}{dx}$

(1) Can be written as $\frac{1}{3}\frac{dz}{dx} + \frac{2z}{3(1+x)} = \frac{x^3}{3} \Rightarrow \frac{dz}{dx} + \frac{2z}{(1+x)} = x^3$

Let $P_1 = \frac{2}{1+x}$ *and* $Q_1 = x^3$

The integrating factor (I.F) = $e^{\int P_1\, dx}$

$$= e^{\int \frac{2}{1+x}\, dx} = e^{2\log(1+x)} = e^{\log(1+x)^2} = (1+x)^2$$

$$\boldsymbol{I\,F} = (1+x)^2$$

$The\ general\ solution: z(I.F) = \int Q_1(I.F)dx + c$

$$y^3\,(1+x)^2 = \int x^3(1+x)^2 dx + c$$

$$= \int x^3(1+2x+x^2)\,\text{dx}$$

$$= \int [x^3 + 2x^4 + x^5]dx = \frac{x^4}{4} + \frac{2x^5}{5} + \frac{x^6}{6} + c$$

Hence the GS is $y^3\,(1+x)^2 = \frac{x^4}{4} + \frac{2x^5}{5} + \frac{x^6}{6} + c$

5. $Solve\ (1-x^2)\frac{dy}{dx} + xy = xy^2$

Solution: The given differential equation is $(1-x^2)\frac{dy}{dx} + xy = xy^2$

Dividing with $(1-x^2)y^2$ on both sides

$$\frac{1}{y^2}\frac{dy}{dx} + \frac{xy}{(1-x^2)y^2} = \frac{xy^2}{(1-x^2)y^2}$$

$$\Rightarrow \frac{1}{y^2}\frac{dy}{dx} + \frac{x}{(1-x^2)y} = \frac{x}{(1-x^2)}$$

Put $\frac{1}{y} = z$ and then $-\frac{1}{y^2}\frac{dy}{dx} = \frac{dz}{dx} \Rightarrow \frac{1}{y^2}\frac{dy}{dx} = -\frac{dz}{dx}$

Can be written as $-\frac{dz}{dx} + \frac{xz}{(1-x^2)} = \frac{x}{(1-x^2)} \Rightarrow \frac{dz}{dx} - \frac{xz}{(1-x^2)} = \frac{-x}{(1-x^2)}$

Let $P_1 = -\frac{x}{(1-x^2)}$ *and* $Q_1 = \frac{-x}{(1-x^2)}$

The integrating factor (I.F) = $e^{\int P_1\, dx}$

$$= e^{\int -\frac{x}{(1-x^2)}\, dx} = e^{\frac{1}{2}\int \frac{-2x}{(1-x^2)}\, dx}$$

$$= e^{\frac{1}{2}\log(1-x^2)} = e^{\log(1-x^2)^{1/2}} = \sqrt{1-x^2} \qquad \boldsymbol{I\,F} = \sqrt{1-x^2}$$

$The\ general\ solution: z(I.F) = \int Q_1(I.F)dx + c$

$$\frac{1}{y}\sqrt{1-x^2} = \int \frac{-x}{(1-x^2)}\sqrt{1-x^2}dx + c$$

$$= \frac{1}{2}\int \frac{-2x}{\sqrt{1-x^2}}dx + c$$

$$= \frac{1}{2}\ 2\sqrt{1-x^2} + c$$

The general solution $\frac{1}{y}\sqrt{1-x^2} = \sqrt{1-x^2} + c$

6.solve $(x+1)\frac{dy}{dx} + 1 = e^{x-y}$

Solution: The given differential equation is $(x+1)\frac{dy}{dx} + 1 = e^{x-y} = e^x e^{-y}$

Dividing with $(x+1)e^{-y}$ on both sides

$$e^y\frac{dy}{dx} + \frac{e^y}{x+1} = \frac{e^x}{x+1}$$

Put $e^y = z$ and then $e^y\frac{dy}{dx} = \frac{dz}{dx}$

Can be written as $\frac{dz}{dx} + \frac{z}{x+1} = \frac{e^x}{x+1}$

Let $P_1 = \frac{1}{x+1}$ *and* $Q_1 = \frac{e^x}{x+1}$

The integrating factor (I.F) = $e^{\int P_1\,dx} = e^{\int \frac{1}{x+1}\,dx} = e^{\log(x+1)} = x+1$

$$\boldsymbol{IF = x+1}$$

$The\ general\ solution: z(I.F) = \int Q_1(I.F)dx + c$

$$e^y(x+1) = \int \frac{e^x}{x+1}(x+1)dx + c = \int (x+1)dx$$

$$= \frac{x^2}{2} + x + c$$

∴ The general solution $e^y(x+1) = \frac{x^2}{2} + x + c$

$7.Solve\ \frac{dy}{dx} + (2x\tan^{-1}y - x^3)(1+y^2) = 0$

Solution: The given differential equation is $\frac{dy}{dx} + (2x\tan^{-1} y - x^3)(1 + y^2) = 0$

Dividing with $1 + y^2$ on both sides

$$\frac{1}{1+y^2}\frac{dy}{dx} + (2x\tan^{-1} y - x^3) = 0$$

$$\frac{1}{1+y^2}\frac{dy}{dx} + 2x\tan^{-1} y = x^3$$

Put $\tan^{-1} y = z$ and then $\frac{1}{1+y^2}\frac{dy}{dx} = \frac{dz}{dx}$

Can be written as $\frac{dz}{dx} + 2xz = x^3$

Let $P_1 = 2x \quad and \;\; Q_1 = x^3$

The integrating factor (I.F) = $e^{\int P_1\, dx} = e^{\int 2x\, dx} = e^{x^2}$ $\qquad$ ***I F*** $= e^{x^2}$

The general solution: $z(I.F) = \int Q_1(I.F)dx + c$

$$\tan^{-1} y\left(e^{x^2}\right) = \int x^3\left(e^{x^2}\right)dx + c$$

$$e^{x^2}\tan^{-1} y = \int x^2\left(e^{x^2}\right)xdx + c$$

let $x^2 = t \Rightarrow 2xdx = dt \Rightarrow xdx = \frac{1}{2}dt$

$$e^{x^2}\tan^{-1} y = \int t\,(e^t)\frac{1}{2}\,dt + c \;=\frac{1}{2}e^t(t-1) + c$$

$\therefore$ The general solution $e^{x^2}\tan^{-1} y = \frac{1}{2}e^{x^2}(x^2 - 1) + c$

8. *Solve* $\frac{dy}{dx}(x^2 y^3 + xy) = 1$

Solution: Given that $\frac{dy}{dx}(x^2 y^3 + xy) = 1$

$\frac{dx}{dy} = x^2 y^3 + xy \Rightarrow \frac{dx}{dy} - xy = x^2 y^3$

dividing with x^2 *on both sides*

$$\frac{1}{x^2}\frac{dx}{dy} - \frac{1}{x}y = y^3 \quad \text{---------(1)}$$

Let $\frac{1}{x} = z \Rightarrow \frac{-1}{x^2}\frac{dx}{dy} = \frac{dz}{dy}$

Equation (1) can be written as $-\frac{dz}{dy} - z\,y = y^3 \Rightarrow \frac{dz}{dy} + z\,y = -y^3$

Let $P_1 = y \quad and \;\; Q_1 = -\,y^3$

The integrating factor (I.F) = $e^{\int P_1\,dx} = e^{\int y\;\;dy} = e^{y^2/2}$

$$\boldsymbol{I\,F = e^{y^2/2}}$$

$The\ general\ solution: z(I.F) = \int Q_1(I.F)dy + c$

$$\frac{1}{x}\left(e^{y^2/2}\right) = \int -y^3\left(e^{y^2/2}\right)dy + c$$

$$\frac{1}{x}\left(e^{y^2/2}\right) = -\int y^2\left(e^{y^2/2}\right)ydy + c$$

let $y^2/2 = t \Rightarrow ydy = dt$

$$\frac{1}{x}\left(e^{y^2/2}\right) = -\int 2t(e^t)dt + c = -2e^t(t-1) + c$$

∴ The general solution $\frac{1}{x}\left(e^{y^2/2}\right) = -2e^{y^2/2}\left(\frac{y^2}{2} - 1\right) + c$

9. $solve\ \frac{dy}{dx} + \frac{y}{x}\log y = \frac{y}{x^2}(\log y)^2$

Solution: Given that $\frac{dy}{dx} + \frac{y}{x}\log y = \frac{y}{x^2}(\log y)^2$

$dividing\ with\ y(\log y)^2\ \ on\ both\ sides$

that $\frac{1}{y(\log y)^2}\frac{dy}{dx} + \frac{y}{xy(\log y)^2}\log y = \frac{y}{x^2 y(\log y)^2}(\log y)^2$

$$\Rightarrow \frac{1}{y(\log y)^2}\frac{dy}{dx} + \frac{1}{x\,log\,y} = \frac{1}{x^2}$$

Let $\frac{1}{log\,y} = z \Rightarrow \frac{-1}{y(\log y)^2}\frac{dy}{dx} = \frac{dz}{dx}$

Equation (1) can be written as $-\frac{dz}{dx} + \frac{z}{x} = \frac{1}{x^2} \Rightarrow \frac{dz}{dx} - \frac{z}{x} = -\frac{1}{x^2}$

Let $P_1 = -\frac{1}{x} \quad and \;\; Q_1 = -\frac{1}{x^2}$

The integrating factor (I.F) = $e^{\int P_1\,dx} = e^{\int -\frac{1}{x}\,dy} = e^{-\log x} = \frac{1}{x}$

$$\boldsymbol{I\,F = \frac{1}{x}}$$

$The\ general\ solution: z(I.F) = \int Q_1(I.F)dx + c$

$$\frac{1}{\log y}\left(\frac{1}{x}\right) = \int -\frac{1}{x^2}\left(\frac{1}{x}\right)dx + c$$

$$\frac{1}{x\log y} = -\int \frac{1}{x^3}dx + c = \frac{1}{2x^2} + c$$

$Hence$ the general solution $\frac{1}{x\log y} = \frac{1}{2x^2} + c$

10.solve $\frac{dy}{dx} = e^{x-y}[e^x - e^y]$

Solution: The given differential equation is

$$\frac{dy}{dx} = e^{x-y}[e^x - e^y] = \frac{dy}{dx} = e^{2x-y} - e^x \Rightarrow \frac{dy}{dx} + e^x = \frac{e^{2x}}{e^y}$$

$multiply\ with\ e^y$ on both sides

$$e^y\frac{dy}{dx} + e^y e^x = e^{2x}$$

Put $e^y = z$ and then $e^y\frac{dy}{dx} = \frac{dz}{dx}$

Can be written as $\frac{dz}{dx} + ze^x = e^{2x}$

Let $P_1 = e^x$ $and\ Q_1 = e^{2x}$

The integrating factor (I.F) = $e^{\int P_1\, dx}$

$$= e^{\int e^x dx} = e^{e^x} \qquad \boldsymbol{I\ F = e^{e^x}}$$

$The\ general\ solution: z(I.F) = \int Q_1(I.F)dx + c$

$e^y e^{e^x} = \int e^{2x}e^{e^x}dx + c = \int e^x e^x e^{e^x}dx + c$ (Let e^x= t ⇒ $e^x dx = dt$)

$$= \int te^t dx + c = e^t(t-1) + c$$

∴ The general solution $e^y e^{e^x} = e^{e^x}(e^x - 1) + c$

3.Exact and Non-Exact Differential Equations:

Definition:

A differential equation $M(x,y)dx + N(x,y)dy \ = 0$ is said to be exact if there exist a function $u(x,y)$ such that $M(x,y)dx + N(x,y)dy = u(x,y)$

In other words, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ and the general solution can be determined by

$$\int M(x,y)dx(y\ treated\ as\ \ constent) + \int N(x,y)dy(Remove\ x - term) = c$$

Problems:

1.$Solve\ (x+2y-3)dy - (2x-y+1)dx = 0$

Solution:$The given differential equation can be written as$

$$(2x - y + 1)dx - (x + 2y - 3)dy = 0$$

Let $M = \ 2x - y + 1$ and $N = \ -(x + 2y - 3)$

Now $\frac{\partial M}{\partial y} = \ -1 \quad and \qquad \frac{\partial N}{\partial x} = \ -1$

$\therefore \ \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ and is Exact

The general solution is

$$\int M(x,y)dx(ytreatedasconstent) + \int N(x,y)dy(Removex - term) = c$$

$$\int(2x - y + 1)dx(ytreatedasconstent) - \int(2y - 3)dy(Removex - term) = c$$

$$\Rightarrow \ 2\frac{x^2}{2} - xy + x \ - 2\frac{y^2}{2} + 3y = c$$

$\Rightarrow$ The general solution $\boldsymbol{x^2 - y^2 - xy + x = c}$

2.$Solve(4x + 3y + 1)dx + (3x + 2y + 1)dy = 0$

Solution:$The given differential equation \ (4x + 3y + 1)dx + (3x + 2y + 1)dy = 0$

Let $M = (4x + 3y + 1)$ and $N = \ (3x + 2y + 1)$

Now $\frac{\partial M}{\partial y} = \ 3 \quad and \frac{\partial N}{\partial x} = \ 3 \qquad \therefore \frac{\partial M}{\partial y} = \ \frac{\partial N}{\partial x}$ and is Exact

The general solution is

$$\int M(x,y)dx(y - treated\ as\ constent) + \int N(x,y)dy(Remove\ x\ term) = c$$

$$\Rightarrow \int (4x + 3y + 1)dx + \int 2y + 1)dy = c$$

$$\Rightarrow 4\ \frac{x^2}{2} + 3xy + x + 2\ \frac{y^2}{2} + y = c$$

$\Rightarrow$The general solution $\boldsymbol{2x^2 + y^2 + 3xy + x + y = c}$

3. Solve $\frac{dy}{dx} = \frac{x^2-4xy-2y^2}{2x^2+4xy-y^2}$

Solution: $Thegivendifferentialequationcanbewrittenas$

$$(2x^2 + 4xy - y^2)dy = (x^2 - 4xy - 2y^2)dx$$

That is $(x^2 - 4xy - 2y^2)dx - (2x^2 + 4xy - y^2)dy = 0$

Let $M = x^2 - 4xy - 2y^2$ and $N = -(2x^2 + 4xy - y^2)$

Now $\frac{\partial M}{\partial y} = -4x - 4y$ $\quad and \frac{\partial N}{\partial x} = -4x - 4y$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$Exactdifferential equation.

The general solution is

$$\int M(x,y)dx(y\ treated\ as\ constent) + \int N(x,y)dy(Remove\ x\ term) = c$$

$$\Rightarrow \int (x^2 - 4xy - 2y^2)dx + \int -(-y^2)dy = c$$

$$\Rightarrow \frac{x^3}{3} - 4\ \frac{x^2}{2}\ y - 2xy^2 + \frac{y^3}{3} = c$$

$$\Rightarrow \frac{x^3}{3} - 2x^2\ y - 2xy^2 + \frac{y^3}{3} = c$$

$\Rightarrow$ The general solution $x^3 - 6x^2\ y - 6xy^2 + y^3 = c$

$4. Solve: (1 + e^{x/y})dx + (1 - \frac{x}{y})e^{x/y}dy = 0$

Solution:$Thegivendifferentialequation$ $(1 + e^{x/y})dx + (1 - \frac{x}{y})e^{x/y}dy = 0$

Let $M = (1 + e^{x/y})$ and $N = (1 - \frac{x}{y})e^{x/y}$

Now $\frac{\partial M}{\partial y} = 0 + e^{\frac{x}{y}}\left(-\frac{x}{y^2}\right) = e^{\frac{x}{y}}\left(-\frac{x}{y^2}\right)$ and

$$\frac{\partial N}{\partial x} = \frac{-1}{y}e^{x/y} + \left(1 - \frac{x}{y}\right)e^{\frac{x}{y}}\left(\frac{1}{y}\right) = -\frac{1}{y}e^{\frac{x}{y}}\left[\,1 - 1 + \frac{x}{y}\right] = e^{\frac{x}{y}}\left(-\frac{x}{y^2}\right)$$

$$\therefore \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \text{and exact}$$

The general solution is

$$\int M(x,y)dx(y\ treated\ as\ constent) + \int N(x,y)dy(Removex\ term) = c$$

$$\Rightarrow \int \left(1 + e^{\frac{x}{y}}\right)dx + 0 = c$$

$$\Rightarrow x + \frac{e^{\frac{x}{y}}}{\frac{1}{y}} = c \quad \Rightarrow \text{The general solution } x + ye^{\frac{x}{y}} = c$$

5. $solve\ (e^y + 1)\cos x\, dx + e^y \sin x\, dy = 0$

Solution: The given differential equation

$$(e^y + 1)\cos x\, dx + e^y \sin x\, dy = 0$$

Let $M = (e^y + 1)Cos\ x$ and $N = e^y sinx$

Now $\frac{\partial M}{\partial y} = e^y Cos\ x$ $\ and\ $ $\frac{\partial N}{\partial x} = e^y\ Cos\ x$ $\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ and exact

The general solution is

$$\int M(x,y)dx(y\ treated\ as\ constent) + \int N(x,y)dy(Removex\ term) = c$$

$$\int (e^y + 1)Cos\ x\ dx \quad = c$$

The general solution $(e^y + 1)sin\ x\ = \text{c}$

6. $solve \frac{dy}{dx} + \frac{ax+hy+g}{hx+by+f} = 0$

Solution: The given differential equation can be written as

$$(hx + by + f)\, dy + (ax + hy + g)dx = 0$$

That is $(ax + hy + g)dx + (hx + by + f)\, dy = 0$

Let $M = ax + hy + g$ and $N = hx + by + f$

Now $\frac{\partial M}{\partial y} = h\ \ and\ \frac{\partial N}{\partial x} = h \qquad \therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}\ \ and\ \ exact$

The general solution is

$$\int M(x,y)dx(y\ treated\ as\ constent) + \int N(x,y)dy(Removex\ term) = c$$

$$\int (ax + hy + g)\,dx + \int (by + f)\,dy = c$$

$$a\frac{x^2}{2} + hxy + gx + b\frac{y^2}{2} + fy = c$$

The general solution $ax^2 + 2hxy + by^2 + 2gx + 2fy = 2c$

4. *Non − exact differential equations*

A differential equation $M(x,y)dx + N(x,y)dy\ = 0$ is said to be non-exact if $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

They are four types of non-exact differential equations

Type-1 Reduce into exact using the Integrating factors.

The following formulae are used to solve such non-exact differential equations.

1. $2xdx + 2ydy = d\,[x^2 + y^2]$
2. $xdy + ydx\ = d(xy)$
3. $\frac{ydx-xdy}{y^2} = d[\ \frac{x}{y}\ ]$
4. $\frac{xdy - ydx}{x^2} = d[\ \frac{y}{x}\ ]$
5. $-\frac{xdy+ydx}{x^2y^2} = d\left[\frac{1}{xy}\right]$
6. $\frac{ydx-xdy}{xy} = d[\log\frac{x}{y}\ ]$
7. $\frac{xdy-ydx}{xy} = d\left[\log\frac{y}{x}\ \right]$
8. $\frac{ydx-xdy}{x^2+y^2} = d[\tan^{-1} x/y\ ]$
9. $\frac{xdy-ydx}{x^2+y^2} = d[\tan^{-1} y/x\ ]$
10. $\frac{ye^xdx-e^xdy}{y^2} = d[\ \frac{e^x}{y}]$
11. $\frac{xe^yy-e^ydx}{x^2} = d[\frac{e^y}{x}]$

Problems:

1. Solve $xdy - ydx = xy^2dx$

Solution: The given differential equation can be written as

$$\frac{xdy-ydx}{y^2} = x\,dx \ \Rightarrow\ \frac{-(\,ydx-xdy)}{y^2} = x\,dx$$

$$\Rightarrow -d\left[\frac{x}{y}\right] = x\,dx$$

$$\Rightarrow\quad -\int d\left[\frac{x}{y}\right] = \int x\,dx + c$$

$$\Rightarrow \frac{-x}{y} = \frac{x^2}{2} + c \quad \Rightarrow \text{The general solution } \frac{x}{y} + \frac{x^2}{2} + c = 0$$

2. $Solve\ (1 + xy)xdy + (1 - xy)ydx = 0$

Solution: The given differential equation can be written as

$$xdy + x^2y\,dy + ydx - xy^2dx = 0$$

$$\Rightarrow x\,dy + y\,dx + xy(xdy - ydx) = 0$$

Dividing with x^2y^2 *on both sides*

$$\Rightarrow \frac{xdy + ydx}{x^2y^2} + \frac{xy(xdy - ydx)}{x^2y^2} = 0$$

$$\Rightarrow \frac{xdy + ydx}{x^2y^2} + \frac{(xdy - ydx)}{xy} = 0$$

$$\Rightarrow d\left[\frac{1}{xy}\right] + d\left[\log \frac{y}{x}\right] = 0 \quad \text{Integrating on both sides}$$

$$\Rightarrow \int d\left[\frac{1}{xy}\right] + \int d\left[\log \frac{y}{x}\right] = c$$

$$\Rightarrow \text{The general solution } \frac{1}{xy} + \log \frac{y}{x} = c$$

3. Solve $xdx + ydy + \frac{xdy - ydx}{x^2 + y^2} = 0$

Solution: Given that $xdx + ydy + \frac{xdy - ydx}{x^2 + y^2} = 0$

$$\Rightarrow \int xdx + \int ydy + \int \frac{xdy - ydx}{x^2 + y^2} = c$$

$$\Rightarrow \int xdx + \int ydy + \int d[\tan^{-1} y/x] = c$$

$$\text{The general solution } \frac{x^2}{2} + \frac{y^2}{2} + \tan^{-1} y/x = c$$

4. $Solve\ \ ydx - xdy + \log x\ dx = 0.$

Solution: Given that $ydx - xdy + \log x\ dx = 0$

Dividing with x^2 on both sides

$$\frac{ydx - xdy}{x^2} + \frac{\log x}{x^2}\,dx = c$$

$$\Rightarrow \int \frac{-(xdy-ydx)}{x^2} + \int \frac{\log x}{x^2}\, dx = c$$

$$\Rightarrow -\int d\left(\frac{y}{x}\right) + \int \log x \, \frac{1}{x^2}\, dx = c$$

$$\Rightarrow -\frac{y}{x} + [\log x \left(-\frac{1}{x}\right) - \int \frac{1}{x}\left(-\frac{1}{x}\right) dx] = c$$

$$\Rightarrow -\frac{y}{x} + [-\frac{1}{x}\log x + \int \frac{1}{x^2}\, dx] = c$$

$$\Rightarrow -\frac{y}{x} - \frac{1}{x}\log x - \frac{1}{x} = c \quad \Rightarrow \text{The general solution } \frac{y}{x} + \frac{1}{x}\log x + \frac{1}{x} = c$$

5. Solve $xdy = [y + x\cos^2\left(\frac{y}{x}\right)]\, d\, x$

Solution: The given differential equation can be written as

$$xdy - ydx = x\cos^2\left(\frac{y}{x}\right) dx$$

$$\Rightarrow \frac{xdy-ydx}{\cos^2\left(\frac{y}{x}\right)} = x\, dx$$

$$\Rightarrow \sec^2\frac{y}{x}\left[\frac{xdy-ydx}{x^2}\right] = \frac{x\, dx}{x^2}$$

$$\Rightarrow \int \sec^2\frac{y}{x}\, d\left(\frac{y}{x}\right) = \int \frac{1}{x}\, dx$$

$\Rightarrow$ The general solution $\tan\frac{y}{x} = \log x + c$

6. $Solve\ (x^2 + y^2 + x)dx - (2x^2 + 2y^2 - y)dy = 0$

Solution: The given differential equation can be written as

$$(x^2 + y^2)dx + xdx - 2(x^2 + y^2) + ydy = 0$$

$$\Rightarrow [dx - 2dy] + xdx + ydy = 0$$

$$\Rightarrow dx - 2dy + \frac{2(xdx+ydy)}{2(x^2+y^2)} = 0$$

$$\Rightarrow \int dx - 2\int \mathrm{dy} + \frac{1}{2}\int \frac{2xdx+2ydy}{(x^2+y^2)} = \mathrm{c}$$

$\Rightarrow$ The general solution $x - 2y + \frac{1}{2}\log(x^2 + y^2) = \mathrm{c}$

Type-2 Homogeneous non-exact Differential equation

A non-exact differential equation is in the form $M(x,y)dx + N(x,y)dy = 0$ where $M(x,y), N(x,y)$ are homogeneous equations the to solve this equation when we reduce into exact using an Integrating Factor $\frac{1}{Mx+Ny}$ $where\ Mx + Ny \neq 0$.

Problems:

1. Solve $x^2y\,dx - (x^3 + y^3)dy = 0$

Solution: In the given differential equation $x^2y\,dx - (x^3 + y^3)dy = 0$

Let M = x^2y N = $-(x^3 + y^3)$

$$\frac{\partial M}{\partial y} = x^2 \qquad \frac{\partial N}{\partial x} = -3x^2 \qquad \therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \text{ not exact}$$

And is homogeneous the integrating factor = $\frac{1}{Mx+Ny} = \frac{-1}{y^4}$

(As $Mx + Ny = (x^2y)x - (x^3 + y^3)y = x^3y - x^3y - y^4 = -y^4$)

Multiply the integrating factor with the given DE

$$-\frac{x^2y}{y^4}dx + \frac{x^3+y^3}{y^4}dy = 0 \Rightarrow -\frac{x^2}{y^3}dx + \left[\frac{x^3}{y^4} + \frac{1}{y}\right]dy = 0$$

Let M = $-\frac{x^2}{y^3}$ $N = \frac{x^3}{y^4} + \frac{1}{y}$

$$\frac{\partial M}{\partial y} = -x^2\left[-\frac{3}{y^4}\right] = \frac{3x^2}{y^4} \qquad \frac{\partial N}{\partial x} = \frac{3x^2}{y^4} + 0 = \frac{3x^2}{y^4}$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \text{exact}$$

The general solution $\int -\frac{x^2}{y^3}dx + \int \frac{1}{y}dy = c$

$$\Rightarrow \frac{-1}{y^3}\frac{x^3}{3} + \log y = c$$

$\Rightarrow$ The general solution $\frac{-x^3}{3y^3} + \log y = c$

2. Solve $y^2\,dx - (x^2 - y^2 - xy)dy = 0$

Solution: In the given differential equation $y^2\,dx + (x^2 - y^2 - xy)dy = 0$

Let M = y^2 N = $(x^2 - y^2 - xy)$

$\frac{\partial M}{\partial y} = 2y$ $\frac{\partial N}{\partial x} = 2x - y$ $\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ not exact

And is homogeneous the integrating factor = $\frac{1}{Mx+Ny} = \frac{1}{y(x^2 - y^2)}$

(As $Mx + Ny = (y^2)x + (x^2 - y^2 - xy)y$

$= xy^2 + x^2y - y^3 - xy^2 = x^2y - y^3$)

$= y(x^2 - y^2)$

Multiply the integrating factor with the given DE

$$\frac{y^2}{y(x^2 - y^2)}dx + \frac{x^2 - y^2 - xy}{y(x^2 - y^2)} dy = 0$$

Let M = $\frac{y}{(x^2 - y^2)}$ $N = \frac{x^2 - y^2 - xy}{y(x^2 - y^2)} = \frac{x^2 - y^2}{y(x^2 - y^2)} - \frac{xy}{y(x^2 - y^2)} = \frac{1}{y} - \frac{x}{x^2 - y^2}$

$\frac{\partial M}{\partial y} = \frac{(x^2 - y^2) - y(0 - 2y)}{(x^2 - y^2)^2} = \frac{(x^2 + y^2)}{(x^2 - y^2)^2}$ $\frac{\partial N}{\partial x} = 0 - \frac{(x^2 - y^2) - x(2x - 0)}{(x^2 - y^2)^2} = \frac{(x^2 + y^2)}{(x^2 - y^2)^2}$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ exact

The general solution $\int \frac{y}{(x^2 - y^2)} dx + \int \frac{1}{y} dy = c$

$y[\frac{1}{2y} \log \frac{x-y}{x+y}] + \log y = c \Rightarrow$ The general solution $\frac{1}{2} \log \frac{x-y}{x+y} + \log y = c$

3.Solve $(x^2y - 2xy^2)dx - (x^3 - 3x^2y)dy = 0$

Solution: In the given differential equation

Let M = $x^2y - 2xy^2$ N = $-(x^3 - 3x^2y)$

$\frac{\partial M}{\partial y} = x^2 - 4xy$ $\frac{\partial N}{\partial x} = -(3x^2 - 6xy)$ $\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ not exact

And is homogeneous the integrating factor = $\frac{1}{Mx+Ny} = \frac{1}{x^2y^2}$

(As $Mx + Ny = (x^2y - 2xy^2)x - (x^3 - 3x^2y)y = x^3y - 2x^2y^2 - x^3y + 3x^2y^2 = x^2y^2$)

Multiply the integrating factor with the given DE

$$\frac{x^2y-2xy^2}{x^2y^2}\,dx - \frac{(x^3-3x^2y}{x^2y^2}\,dy = 0$$

$$[\frac{1}{y}-\frac{2}{x}]\,d\,x - [\frac{x}{y^2}-\frac{3}{y}]dy = 0$$

Let M = $\frac{1}{y}-\frac{2}{x}$ $\quad N = -\frac{x}{y^2}+\frac{3}{y}$

$\frac{\partial M}{\partial y} = -\frac{1}{y^2}-0=-\frac{1}{y^2}$ $\quad \frac{\partial N}{\partial x} = -\frac{1}{y^2}-0=-\frac{1}{y^2}$

$\therefore \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$ exact

The general solution $\int[\frac{1}{y}-\frac{2}{x}]$ dx $-\int\frac{-3}{y}dy = c$

$\Rightarrow$ The general solution $\frac{x}{y}-2\log x+3\log y=c$

Type-3 $\boldsymbol{Non-exact\ yf(x,y)dx+xf(x,y)dy=0\ form}$

A non-exact differential equation is in the form $M(x,y)dx+N(x,y)dy\ =0$ or

$yf(x,y)dx+xf(x,y)dy=0$ the to solve this equation when we reduce into exact using an Integrating Factor $\frac{1}{Mx-Ny}$ $where\ Mx-Ny\neq 0.$

$Problems$: 1.Solve $y(1+xy)dx+x(1-xy)dy=0$

$Solution$: $In\ the\ given\ differential\ equation$

M = $y(1+xy)=y+xy^2$ $\quad$ N = $x(1-xy)=x-x^2y$

$\frac{\partial M}{\partial y}$ = $1+2xy$ $\quad \frac{\partial N}{\partial x}=1-2xy$ $\quad \therefore\ \frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$ not exact.

$The\ integrating\ factor = \frac{1}{Mx-Ny}=\frac{1}{2\,x^2y^2}$

$(\,since\ \ Mx-Ny=(y+xy^2)x-(x-x^2y)y=xy+x^2y^2-xy+\ x^2y^2=2\,x^2y^2\ \neq 0)$

Multiply the integrating factor with the given DE

$$\frac{y+xy^2}{2\,x^2y^2}\,dx+\frac{x-x^2y}{2\,x^2y^2}\,dy=0$$

$$[\frac{1}{2yx^2}+\frac{1}{2x}]\,dx + [\frac{1}{2xy^2}-\frac{1}{2y}]dy = 0$$

Let M = $\frac{1}{2yx^2} + \frac{1}{2x}$ N = $\frac{1}{2xy^2} - \frac{1}{2y}$

$\frac{\partial M}{\partial y} = -\frac{1}{2y^2x^2}$ $\frac{\partial N}{\partial x} = -\frac{1}{2y^2x^2}$ $\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ exact

The general solution $\int \frac{1}{2yx^2} + \frac{1}{2x}\ dx + \int \frac{1}{2xy^2} - \frac{1}{2y}\ dy = c$

$\Rightarrow$ The general solution $\frac{-1}{2xy} + \frac{1}{2}\log x - \frac{1}{2}\log y = c$

2. Solve $y(xy + 2x^2y^2)dx + x(xy - x^2y^2)dy = 0$

Solution: *In the given differential equation*

M = $y(xy + 2x^2y^2) = xy^2 + 2x^2y^3$ N = $x(xy - x^2y^2) = x^2y - x^3y^2$

$\frac{\partial M}{\partial y}$ = $2xy + 6x^2y^2$ $\frac{\partial N}{\partial x} = 2xy - 3x^2y^2$ $\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ not exact.

$The\ integrating\ factor = \frac{1}{Mx - Ny} = \frac{1}{3x^3y^3}$

$(\ since\ Mx - Ny = (xy^2 + 2x^2y^3)x - (x^2y - x^3y^2)y$

$= x^2y^2 + 2x^3y^3 - x^2y^2 + x^3y^3 = 3x^3y^3 \neq 0)$

Multiply the integrating factor with the given DE

$$\frac{xy^2 + 2x^2y^3}{3x^3y^3}\ dx + \frac{x^2y - x^3y^2}{3x^3y^3}\ dy = 0$$

$$\left[\frac{1}{3x^2y} + \frac{2}{3x}\right] dx + \left[\frac{1}{3xy^2} - \frac{1}{3y}\right] dy = 0$$

Let M = $\frac{1}{3x^2y} + \frac{2}{3x}$ N = $\frac{1}{3xy^2} - \frac{1}{3y}$

$\frac{\partial M}{\partial y} = -\frac{1}{3y^2x^2}$ $\frac{\partial N}{\partial x} = -\frac{1}{3y^2x^2}$ $\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ exact

The general solution $\int \frac{1}{3x^2y} + \frac{2}{3x}\ dx + \int \frac{1}{3xy^2} - \frac{1}{3y}\ dy = c$

$\Rightarrow$ The general solution $\frac{-1}{3xy} + \frac{2}{3}\log x - \frac{1}{3}\log y = c$

3..*Solve* $x(1 + xy)dy + y(1 - xy)dx = 0$

Solution: *The given DE can be written as* $y(1 - xy)dx + x(1 + xy)dy = 0$

$In\ the\ given\ differential\ equation$

M = $y(1-xy) = y - xy^2$ N == $x(1+xy) = x + yx^2$

$\frac{\partial M}{\partial y} = 1 - 2xy$ $\frac{\partial N}{\partial x} = 1 + 2xy$ $\therefore$ $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ not exact.

$The\ integrating\ factor = \frac{1}{Mx-Ny} = \frac{-1}{2\,x^2y^2}$

$(\ since\ \ Mx - Ny = (y-xy^2)x - (x+x^2y)y = xy - x^2y^2 - xy - \ x^2y^2 = -\,2\,x^2y^2\ \ \neq 0)$

Multiply the integrating factor with the given DE

$$\frac{y-xy^2}{-2\,x^2y^2}\,dx + \frac{x+x^2y}{-2\,x^2y^2}\,dy = 0$$

$$[\ \frac{1}{2yx^2} - \frac{1}{2x}\ ]\,dx + \ [\ \frac{1}{2xy^2} + \frac{1}{2y}\ ]dy = 0$$

Let M = $\frac{1}{2yx^2} - \frac{1}{2x}$ N = $\frac{1}{2xy^2} + \frac{1}{2y}$

$\frac{\partial M}{\partial y} = -\frac{1}{2y^2x^2}$ $\frac{\partial N}{\partial x} = -\frac{1}{2y^2x^2}$ $\therefore$ $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ exact

The general solution $\int \frac{1}{2yx^2} - \frac{1}{2x}\ \ dx + \ \int \frac{1}{2xy^2} + \frac{1}{2y}\ dy = c$

$\Rightarrow$ The general solution $\frac{-1}{2xy} - \ \frac{1}{2}\ \log x + \frac{1}{2}\log y = c$

Type-4 $\boldsymbol{Non - exact\ Mdx + Ndy = 0\ form}$

A non-exact differential equation is in the form $M(x,y)dx + N(x,y)dy\ \ = 0$ or

$M(x,y)dx + N(x,y)(least\ function\ copare\ to\ M)dy\ \ = 0$ the to solve this equation when we reduce into exact using an Integrating Factor $e^{\int f(x)dx}$ where

$$f(x) = \frac{1}{N}\ [\ \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\ ]$$

Problems: 1. Solve $2xydy - (x^2 + y^2 + 1)dx = 0$

Solution: The given differential equation an be written as

$$(x^2 + y^2 + 1)dx - 2xydy = 0$$

Let M = $x^2 + y^2 + 1$ $N = \ -2xy$

$\frac{\partial M}{\partial y} = 2y$ $\frac{\partial N}{\partial x} = -2y \Rightarrow \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ $non\ exact\ DE$

Clearly N is a least function $\therefore f(x) = \frac{1}{N}\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right] = \frac{1}{-2xy}[2y + 2y] = \frac{1}{-2xy}[4y] = \frac{-2}{x}$

Integrating factor, I F = $e^{\int f(x)dx} = e^{\int -\frac{2}{x}dx} = e^{-2\log x} = e^{\log x^{-2}} = \frac{1}{x^2}$

I F = $\frac{1}{x^2}$

Multiply the IF with the given DE we get

$$\left[\frac{x^2+y^2+1}{x^2}\right]dx - \frac{2xy}{x^2}dy = 0$$

$$\left[1 + \frac{y^2}{x^2} + \frac{1}{x^2}\right]dx - \frac{2y}{x}dy = 0$$

Let $M_1 = 1 + \frac{y^2}{x^2} + \frac{1}{x^2}$ $N_1 = -\frac{2y}{x}$

$\frac{\partial M_1}{\partial y} = 0 + \frac{2y}{x^2} + 0 = \frac{2y}{x^2}$ $\frac{\partial N_1}{\partial x} = -2y\left(-\frac{1}{x^2}\right) = \frac{2y}{x^2}$ $\therefore \frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$ $exact\ DE$

The general solution is $\int\left[1 + \frac{y^2}{x^2} + \frac{1}{x^2}\right]dx + 0 = c$

$\Rightarrow x - \frac{y^2}{x} - \frac{1}{x} = c$ $\Rightarrow$ The general solution $x^2 - y^2 - 1 = cx$

Problems: 2. Solve $(x^2 + y^2 + 2x)dx + 2ydy = 0$

Solution: The given differential equation is $(x^2 + y^2 + 2x)dx + 2ydy = 0$

Let M = $x^2 + y^2 + 2x$ $N = 2y$

$\frac{\partial M}{\partial y} = 2y$ $\frac{\partial N}{\partial x} = 0 \Rightarrow \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ $non\ exact\ DE$

Clearly N is a least function $\therefore f(x) = \frac{1}{N}\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right] = \frac{1}{2y}[2y - 0] = 1$

Integrating factor, I F = $e^{\int f(x)dx} = e^{\int 1\,dx} = e^x$

I F = e^x

Multiply the IF with the given DE we get

$$e^x(x^2 + y^2 + 2x)dx + 2ye^x dy = 0$$

Let $M_1 = e^x(x^2+y^2+2x)$ $\quad N_1 = 2ye^x$

$\frac{\partial M_1}{\partial y} = e^x(2y) = 2ye^x \qquad \frac{\partial N_1}{\partial x} = 2ye^x \qquad \therefore \frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$ $\quad exact\ DE$

The general solution is $\int e^x(x^2+y^2+2x)dx + 0 = c$ using Integration by parts

$$e^x(x^2+y^2+2x) - e^x(2x+2) + e^x(2) = c$$

$$e^x(x^2+y^2+2x-2x-2+2) = c$$

$\Rightarrow$ The general solution $e^x(x^2+y^2) = c$

Problems: 3. Solve $(1+y+x^2y)dx + (x+x^3)dy = 0$

Solution: The given differential equation is $(1+y+x^2y)dx + (x+x^3)dy = 0$

Let M = $1+y+x^2y$ $\quad N = x+x^3$

$\frac{\partial M}{\partial y} = 1+x^2 \qquad \frac{\partial N}{\partial x} = 1+3x^2 \Rightarrow \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ $non\ exact\ DE$

Clearly N is a least function $\quad \therefore\ f(x) = \frac{1}{N}\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right] = \frac{1}{x+x^3}\left[1+x^2-1-3x^2\right]$

$$= \frac{-2x^2}{x(1+x^2)} = \frac{-2x}{(1+x^2)}$$

Integrating factor, I F = $e^{\int f(x)dx} = e^{\int \frac{-2x}{(1+x^2)}dx} = e^{-\log(1+x^2)}$

$$\text{I F} = \frac{1}{1+x^2}$$

Multiply the IF with the given DE we get

$$\frac{1+y+x^2y}{1+x^2}dx + \frac{x+x^3}{1+x^2}dy = 0 \Rightarrow \left(\frac{1}{1+x^2}+y\right)dx + xdy = 0$$

Let $M_1 = \frac{1}{1+x^2}+y$ $\quad N_1 = x$

$\frac{\partial M_1}{\partial y} = 1 \qquad \frac{\partial N_1}{\partial x} = 1 \qquad \therefore \frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$ $\quad exact\ DE$

The general solution is $\int \frac{1}{1+x^2}+y\ dx + 0 = c$ using Integration by parts

$\Rightarrow$ The general solution $\tan^{-1}x + xy = c$

Type-5 $Non-exact\ Mdx+Ndy=0\ form$

A non-exact differential equation is in the form $M(x,y)dx+N(x,y)dy\ =0$ or

$M(x,y)(least\ function\ copare\ to\ N)dx+N(x,y)dy\ =0$ the to solve this equation when we reduce into exact using an Integrating Factor $e^{\int g(y)dy}$ where

$$g(y)=\frac{1}{M}\left[\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right]$$

Problems: 1. Solve $(xy^3+y)dx+2(x^2y^2+x+y^4)dy\ =\ 0$

Solution: The given differential equation is $(xy^3+y)dx+2(x^2y^2+x+y^4)dy\ =\ 0$

Let M = xy^3+y $\qquad N=2(x^2y^2+x+y^4)$

$\frac{\partial M}{\partial y}=1+3xy^2 \qquad \frac{\partial N}{\partial x}=2(2xy^2+1)=\ 4xy^2+2\ \Rightarrow \frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}\ non\ exact\ DE$

Clearly M is a least function $\quad \therefore g(y)=\ \frac{1}{M}\left[\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right]=\frac{1}{xy^3+y}\ [4xy^2+2-1-3xy^2\ ]$

$$=\frac{1}{y(xy^2+1)}\{xy^2+1\}=\frac{1}{y}$$

$I\ F=\ e^{\int g(y)dy}\ =\ e^{\int\frac{1}{y}dy}=e^{\log y}=y \qquad$ I F = y

Multiply the IF with the given DE we get

$$(xy^4+y^2)dx+2(x^2y^3+xy+y^5)dy\ =\ 0$$

Let $M_1=\ xy^4+y^2 \qquad N_1=2(x^2y^3+xy+y^5)$

$\frac{\partial M_1}{\partial y}=\ 4xy^3+2y \qquad \frac{\partial N_1}{\partial x}=4xy^3+2y \qquad \therefore\frac{\partial M_1}{\partial y}=\frac{\partial N_1}{\partial x} \quad exact\ DE$

The general solution is

$$\int xy^4+y^2\ dx+\int 2\,y^5\ \ dy+0=c$$

$$\frac{x^2y^4}{2}+xy^2+2\ \frac{y^6}{6}=c$$

$\Rightarrow$ The general solution $3x^2y^4+6xy^2+2y^6=c$

2.Solve $(y^4 + 2y)dx + (xy^3 - 4x + 2y^4)dy = 0$

Solution: The given differential equation is $(y^4 + 2y)dx + (xy^3 - 4x + 2y^4)dy = 0$

Let M = $y^4 + 2y$ $\quad N = xy^3 - 4x + 2y^4$

$$\frac{\partial M}{\partial y} = 4y^3 + 2 \qquad \frac{\partial N}{\partial x} = (y^3 - 4 + 0) = y^3 - 4 \Rightarrow \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \; non\ exact\ DE$$

Clearly M is a least function $\quad \therefore g(y) = \frac{1}{M}\left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right]$

$$= \frac{1}{y^4+2y}\,[\,y^3 - 4 - 4y^3 - 2]$$

$$= \frac{1}{y(y^3+2)}\{-3y^3 - 6\} = \frac{-3(y^3+2)}{y(y^3+2)} = \frac{-3}{y}$$

$$I\,F = e^{\int g(y)dy} = e^{\int \frac{-3}{y}dy} = e^{-3\log y} = \frac{1}{y^3} \qquad \text{I F} = \frac{1}{y^3}$$

Multiply the IF with the given DE we get

$$\frac{y^4+2y}{y^3}\,dx + \frac{xy^3-4x+2y^4}{y^3}\,dy = 0 \Rightarrow \left[y + \frac{2}{y^2}\right]dx + [x - \frac{4x}{y^3} + 2y]dy = 0$$

Let $M_1 = y + \frac{2}{y^2} \quad N_1 = x - \frac{4x}{y^3} + 2y$

$$\frac{\partial M_1}{\partial y} = 1 - \frac{4}{y^3} \qquad \frac{\partial N_1}{\partial x} = 1 - \frac{4}{y^3} \qquad \therefore \frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x} \quad exact\ DE$$

The general solution is $\quad \int[\,y + \frac{2}{y^2}\,]\,dx + \int 2y\ \ dy = c \Rightarrow xy + \frac{2x}{y^2} + y^2 = c$

The general solution $xy + \frac{2x}{y^2} + y^2 = c$

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GOVERNMENT DEGREE COLLEGE, RAVULAPALEM
NAAC Accredited with 'B' Grade(2.61 CGPA)
(Affiliated to Adikavi Nannaya University)
Beside NH-16, Main Road, Ravulapalem-533238, Dr.B.R.Ambedkar Dist., A.P, INDIA
E-Mail : jkcjyec.ravulapalem@gmail.com, Phone : 08855-257061
ISO 50001:2011, ISO 14001:2015, ISO 9001:2015 Certified College

UNIT – II

. First order but not first degree differential equations

(Solvable for p, x and y)

-B. Srinivasa Rao. Maths GDC RVPM

To solve the differential equation $f(x, y, p) = 0$ *where* $\boldsymbol{p} = \frac{dy}{dx}$ and is first order but not first degree.

This type of differential equation can be solved from the following methods

1.Solveble for p.

2. Solvable for y.

3. Solvable for x.

4.Clairaut's form of Differential equation.

Model-1. Equations for solvable for p: In the differential equation $f(x, y, p) = 0$ is first order but not first

Degree and x, y *are more than first degree then it is solveble for* p.

Problems:

1.Solve $p^2 - 7p + 12 = 0$

Solution: Given that $p^2 - 7p + 12 = 0$

$$1\times 12 = 12 = 4 \times 3 = -4 \times -3$$

$$\Rightarrow p^2 - 4p - 3p + 12 = 0 \quad \Rightarrow \quad p(p-4) - 3(p-4) = 0$$

$$\Rightarrow (p-3)(p-4) = 0$$

$$\Rightarrow p = 3 \text{ or } p = 4$$

$$If\ p = 3 \Rightarrow \frac{dy}{dx} = 3 \ \Rightarrow dy = 3dx$$

Integrating both sides

$$\int dy = \int 3\, dx \Rightarrow y = 3x + c \Rightarrow y - 3x - c = 0$$

$If\ p = 4 \Rightarrow \frac{dy}{dx} = 4 \Rightarrow dy = 4dx$

Integrating both sides

$\int dy = \int 4\, dx \Rightarrow y = 4x + c \Rightarrow y - 4x - c = 0$

The general solution of the differential equation is $(y - 3x - c)(y - 4x - c) = 0$

2.Solve $p^2 - 5p + 6 = 0$

Solution: Given that $p^2 - 5p + 6 = 0$

$1 \times 6 = 6 = 2 \times 3 = -2 \times -3$

$\Rightarrow p^2 - 2p - 3p + 6 = 0 \Rightarrow p(p-2) - 3(p-2) = 0$

$\Rightarrow (p-2)(p-3) = 0$

$\Rightarrow p = 2\ or\ p = 3$

$If\ p = 2 \Rightarrow \frac{dy}{dx} = 2 \Rightarrow dy = 2dx$

Integrating both sides

$\int dy = \int 2\, dx \Rightarrow y = 3x + c \Rightarrow y - 2x - c = 0$

$If\ p = 3 \Rightarrow \frac{dy}{dx} = 3 \Rightarrow dy = 3dx$

Integrating both sides

$\int dy = \int 3\, dx \Rightarrow y = 3x + c \Rightarrow y - 3x - c = 0$

The general solution of the differential equation is $(y - 2x - c)(y - 3x - c) = 0$

3.Solve $4y^2p^2 + 2xy(3x+1)p + 3x^3 = 0$

Solution: Given that $4y^2p^2 + 2xy(3x+1)p + 3x^3 = 0$

$\Rightarrow 4y^2p^2 + 6x^2yp + 2xyp + 3x^3 = 0$

$\Rightarrow 2yp(2yp + 3x^2) + x(2yp + 3x^2) = 0$

$\Rightarrow (2yp + 3x^2)(2yp + x) = 0$

$\Rightarrow (2yp + 3x^2) = 0\ or\ (2yp + x) = 0$

$\Rightarrow p = -\frac{3x^2}{2y}$ or $p = -\frac{x}{2y}$

$\Rightarrow \frac{dy}{dx} = -\frac{3x^2}{2y}$ or $\frac{dy}{dx} = -\frac{x}{2y}$

$\Rightarrow 2ydy = -3x^2dx\ \ or\ 2ydy = -xdx$ integrating on bot sides

$\int 2ydy = -\int 3x^2dx\ \ or\ \int 2ydy = -\int xdx$

$$\Rightarrow \frac{2y^2}{2} = -\frac{3x^3}{3} + c \quad or \ \frac{2y^2}{2} = -\frac{x^2}{2} + c$$

$$\Rightarrow y^2 + x^3 - c = 0 \ or \ 2y^2 + x^2 - 2c = 0$$

The general solution is $(y^2 + x^3 - c)(2y^2 + x^2 - 2c) = 0$

** 4. Solve $xyp^2 + (3x^2 - 2y^2)p - 6xy = 0 = 0$

Solution: Given that $xyp^2 + (3x^2 - 2y^2)p - 6xy = 0 = 0$

$$\Rightarrow xyp^2 + 3x^2p - 2y^2p - 6xy = 0 = 0$$

$$\Rightarrow xp(yp + 3x) - 2y(yp + 3x) = 0$$

$$\Rightarrow (yp + 3x)(xp - 2y) = 0$$

$$\Rightarrow yp + 3x = 0 \ or \ xp - 2y$$

$$\Rightarrow p = -\frac{3x}{y} \text{ or } p = \frac{2y}{x}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{3x}{y} \text{ or } \frac{dy}{dx} = \frac{2y}{x}$$

$\Rightarrow ydy = -3xdx \quad or \ \frac{dy}{y} = \frac{2}{x}dx$ integrating on bot sides

$$\int ydy = -\int 3xdx \quad or \ \int \frac{dy}{y} = \int \frac{2}{x}dx$$

$$\Rightarrow \frac{y^2}{2} = -\frac{3x^2}{2} + c \quad or \ \log y = 2\log x + log\ c$$

$$\Rightarrow y^2 + 3x^2 - 2c = 0 \ \ or \ \ \log y = \log x^2c$$

$$\Rightarrow y^2 + 3x^2 - 2c = 0 \ or \ \ y - cx^2 = 0$$

The general solution is $(y^2 + 3x^2 - 2c)(y - cx^2) = 0$

5.solve $x^2p^2 + 3xyp + 2y^2 = 0$

Solution: Given that $x^2p^2 + 3xyp + 2y^2 = 0$

$$\Rightarrow x^2p^2 + xyp + 2xyp + 2y^2 = 0$$

$$\Rightarrow xp(xp + y) + 2y(xp + y) = 0$$

$$\Rightarrow (xp + 2y)(xp + y) = 0$$

$$\Rightarrow (xp + 2y) = 0, \ (xp + y) = 0$$

$$\Rightarrow xp + 2y = 0 \ or \ xp + y = 0$$

$$\Rightarrow p = -\frac{2y}{x} \quad or \ p = -\frac{y}{x} \quad \Rightarrow \frac{dy}{dx} = -\frac{2y}{x} \quad or \ \ \frac{dy}{dx} = -\frac{y}{x}$$

$\Rightarrow \frac{dy}{y} = -\frac{2dx}{x} \quad or \ \ \frac{dy}{y} = -\frac{dx}{x}$ integrating both sides

$$\Rightarrow \int \frac{dy}{y} = -\int \frac{2dx}{x} + c \quad \text{or} \int \frac{dy}{y} = -\int \frac{dx}{x} + c$$

$$\Rightarrow \log y = -2\log x + \log c \qquad or \ \log y = -\log x + \log c$$

$$\Rightarrow \log y = \log c/x^2 \ \text{ or } \log y = \log \frac{c}{x} \Rightarrow x^2y - c = 0 \ or \ xy - c = 0$$

The general solution $(x^2y - c)(xy - c)$

6.Solve $xy^2(p^2 + 2) = 2py^3 + x^3$

Solution: Given that $xy^2(p^2 + 2) = 2py^3 + x^3$

$$\Rightarrow xy^2p^2 + 2xy^2 = 2py^3 + x^3$$

$$\Rightarrow xy^2p^2 - 2py^3 + (2xy^2 - x^3) = 0 \text{ is a quadratic in p}$$

$$p = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$p = \frac{2y^3 \pm \sqrt{(2y^3)^2 - 4(xy^2)(2xy^2 - x^3)}}{2(xy^2)}$$

$$= \frac{-2y^3 \pm \sqrt{4y^6 - 8x^2y^4 + 4x^4y^2)}}{2(xy^2)}$$

$$= \frac{2y^3 \pm \sqrt{4y^2(y^4 - 2x^2y^2 + x^4)}}{2(xy^2)}$$

$$\text{p} = \frac{2y^3 \pm 2y\sqrt{(x^2 - y^2)^2}}{2xy^2} = \frac{y^2 \pm [x^2 - y^2]}{xy}$$

$$\frac{dy}{dx} = \frac{y^2 + [x^2 - y^2]}{xy} \ or \ \frac{dy}{dx} = \frac{y^2 - [x^2 - y^2]}{xy}$$

$$\frac{dy}{dx} = \frac{x^2}{xy} - - - (1) \qquad or \ \frac{dy}{dx} = \frac{2y^2 - x^2}{xy} - - - (2)$$

If $\frac{dy}{dx} = \frac{x^2}{xy} \Rightarrow ydy = xdx \Rightarrow \int ydy = \int xdx \Rightarrow \frac{y^2}{2} = \frac{x^2}{2} + \frac{c}{2} \Rightarrow x^2 - y^2 + c = 0$

$If \ \frac{dy}{dx} = \frac{2y^2 - x^2}{xy}$ is homogeneous differential equation

Put $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$$\therefore \ v + x\frac{dv}{dx} = \frac{2(vx)^2 - x^2}{x(vx)} = \frac{2v^2 - 1}{v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{2v^2 - 1}{v} - v = \frac{v^2 - 1}{v}$$

$\Rightarrow \frac{v}{v^2 - 1}dv = \frac{dx}{x}$ integrating on both sides

$\int \frac{v}{v^2-1} dv = \int \frac{dx}{x} + c \Rightarrow \frac{1}{2}\log(v^2-1) = \log\ x + logc$

$\Rightarrow \log(v^2-1) = 2\log\ x + 2logc = \log\ x^2c^2$

$\Rightarrow \log\left(\frac{y^2}{x^2} - 1\right) = \log\ x^2c^2$

$\Rightarrow\ y^2 - x^2 = x^4c^2 \Rightarrow y^2 - x^2 - x^4c^2 = 0$

The general solution $(x^2 - y^2 + c)(y^2 - x^2 - x^4c^2) = 0$

** 7. Solve $p^2 + 2pycot\ x = y^2$

Solution: Given that $p^2 + 2pycot\ x = y^2$

$\Rightarrow p^2 + 2pycot\ x + y^2cot^2x = y^2 + y^2cot^2x$

$\Rightarrow (p + y\cot x)^2 = y^2(1 + cot^2x) = y^2Cosc^2x$

$\Rightarrow\ p + y\cot x = \pm(ycosc\ x)$

$\Rightarrow\ p + y\cot x = (ycosc\ x)\ \ or\ \ \Rightarrow\ p + y\cot x = -(ycosc\ x)$

$\Rightarrow\ p + y(\cot x - cosc\ x) = 0\ \ or\ \ p + y(\cot x + cosc\ x) = 0$

If $p + y(\cot x - cosc\ x) = 0\ \ \Rightarrow \frac{dy}{dx} = -\ y(\cot x - cosc\ x)$

$\frac{dy}{y} = -(\cot x - cosc\ x)dx$ Integrating both sides

$\int \frac{dy}{y} = -\int(\frac{\cos x - 1}{\sin x})dx \Rightarrow \log y = \int \frac{1-\cos x}{\sin x}$ dx

$= \int \frac{2\ sin^2 x/2}{2\sin\frac{x}{2}\cos\frac{x}{2}}\ dx = \int Tan\frac{x}{2}\ dx = 2\log\ (secx/2)$

Log y = log $(sec^2x/2) + \log\ c = \log\ (csec^2x/2)$

$\therefore (y - csec^2x/2) = 0$

If $p + y(\cot x + cosc\ x) = 0\ \ \Rightarrow \frac{dy}{dx} = -\ y(\cot x + cosc\ x)$

$\frac{dy}{y} = -(\cot x + cosc\ x)dx$ Integrating both sides

$\int \frac{dy}{y} = -\int \frac{1+\cos x}{\sin x}\ dx \Rightarrow \log y = -\int \frac{2cos^2x/2}{2\sin\frac{x}{2}\cos\frac{x}{2}}$ dx

$= -\int \frac{\cos\frac{x}{2}}{\sin\frac{x}{2}}\ dx = -\int cot\frac{x}{2}\ dx = -2\log\ (\sin x/2)$

Log y = -log $(sin^2x/2) + \log\ c = \log\ (ccosc^2x/2)$

$\therefore (y - ccosc^2x/2) = 0$

The general solution is $(y - csec^2x/2)(y - ccosc^2x/2) = 0$

Model-2 : Equations for solvable for y

$Working\ method\ for\ solvable\ for\ y$ Suppose the given differential

equation is in the form $f(x, y, p) = 0$ where y is of first degree and is in one time in the overall differentia$n$

First the given differential equation can be written as $y = f(x, p)$ and then to differentiate wrt x on both sides replace $\frac{dy}{dx}$ by p $and\ solve.$

Problems: 1. Solve $y + px = p^2x^4$

Solution: Given that $y + px = p^2x^4$

$y = -px + p^2x^4$ Differentiate wrt x on both sides

$$\frac{dy}{dx} = -\left[p + x\frac{dp}{dx}\right] + [2p\frac{dp}{dx}\ x^4 + p^2(4x^3)]$$

$\Rightarrow p = x\frac{dp}{dx}\ [\ 2x^3p - 1] + [4x^3p^2 - p]$

$\Rightarrow x\frac{dp}{dx}\ [\ 2x^3p - 1] + [4x^3p^2 - 2p] = 0$

$\Rightarrow x\frac{dp}{dx}\ [\ 2x^3p - 1] + 2p[\ 2x^3p - 1] = 0$

$\Rightarrow [\ 2x^3p - 1]\left[x\frac{dp}{dx} + 2p\right] = 0 \Rightarrow 2x^3p - 1 = 0\ or\ x\frac{dp}{dx} + 2p = 0$

If $2x^3p - 1 = 0\ we\ get\ a\ sigular\ solution.$

$\therefore x\frac{dp}{dx} + 2p = 0\ \Rightarrow x\frac{dp}{dx} = -2p \Rightarrow \frac{dp}{p} = \frac{-2dx}{x}$

Integrating both sides

$\log p = -2\log x + \log c = \log\frac{c}{x^2}\ \Rightarrow p = \frac{c}{x^2}$ put the value in the given differential equation

$y + px = p^2x^4\ we\ get\ y + \frac{c}{x^2}\ x = [\ \frac{c}{x^2}\ ]^2x^4 \Rightarrow xy = c^2x - c$

2.Solve $y = xp^2 + p$

Solution: Given that $y = xp^2 + p$

Differentiate wrt x on both sides

$$\frac{dy}{dx} = \left[p^2 + 2xp\frac{dp}{dx}\right] + \frac{dp}{dx}$$

$\Rightarrow p = p^2 + (2xp + 1)\frac{dp}{dx} \Rightarrow (2xp + 1)\frac{dp}{dx} + p(p - 1) = 0$

$\Rightarrow p(p - 1)\frac{dx}{dp} + (2xp + 1) = 0 \Rightarrow \frac{dx}{dp} + \frac{2xp}{p(p-1)} = \frac{-1}{p(p-1)}$

Is a linear differential equation in $\frac{dx}{dp}$

Let P = $\frac{2}{(p-1)}$ $\quad Q = \frac{-1}{p(p-1)}$

I F $= e^{\int P\,dp} = e^{\int \frac{2}{(p-1)}dp} = e^{2\log(p-1)} = (p-1)^2$

The general solution

$x(IF) = \int Q(IF)dp + c$

$$\Rightarrow x(p-1)^2 = \int \frac{-1}{p(p-1)}(p-1)^2 dp + c = -\int \frac{(p-1)}{p}dp$$

$$\Rightarrow x(p-1)^2 = \int [\frac{1}{p} - 1\,]dp = \log\ p - p + c$$

The general solution is $x(p-1)^2 = \log\ p - p + c$

3. Solve $y = 2px + x^2p^4$

Solution: Given that $y = 2px + x^2p^4$

Differentiate wrt x on both sides

$\frac{dy}{dx} = 2\left[p + x\frac{dp}{dx}\right] + [4p^3\frac{dp}{dx}\ x^2 + p^4(2x)]$

$\Rightarrow p = 2x\frac{dp}{dx}\ [\,1 + 2xp^3] + 2p + 2xp^4$

$\Rightarrow 2x\frac{dp}{dx}\ [1 + 2xp^3\,] + [2xp^4 + p] = 0$

$\Rightarrow 2x\frac{dp}{dx}\ [1 + 2xp^3\,] + p[\,1 + 2xp^3] = 0$

$\Rightarrow [1 + 2xp^3\,]\left[2x\frac{dp}{dx} + p\right] = 0 \Rightarrow 1 + 2xp^3 = 0\ or\ 2x\frac{dp}{dx} + p = 0$

If $1 + 2xp^3 = 0$ *we get a sigular solution.*

$\therefore 2x\frac{dp}{dx} + p = 0 \Rightarrow 2x\frac{dp}{dx} = -p \Rightarrow 2\ \frac{dp}{p} = \frac{-dx}{x}$

Integrating both sides

$2\log p = -\log x + \log c = \log\frac{c}{x} \Rightarrow p^2 = \frac{c}{x}$ put the value in the given differential equation

$$y = 2px + x^2p^4\ we\ get\ 2px = y - x^2p^4$$

Squaring on both sides $4p^2x^2 = (y - x^2p^4)^2 \Rightarrow 4\left(\frac{c}{x}\right)x^2 = [\,y - x^2\left(\frac{c}{x}\right)^2]^2$

$$\Rightarrow 4cx = [y - c^2]^2$$

Model-3: Equations for solvable for x

$Working\ method\ for\ solvable\ for\ x$ Suppose the given differential equation is in the form $f(x,y,p)=0$ where x is of first degree and is in one time in the overall $differentian$

First the given differential equation can be written as $x=f(y,p)$ and then to differentiate wrt x on both sides replace $\frac{dx}{dy}$ by $\frac{1}{p}$ $and\ solve.$

Problems:

1. Solve $y^2\log y = xyp+p^2$

Solution: Given that $y^2\log y = xyp+p^2$

$$\Rightarrow x=\frac{y^2\log y-p^2}{yp}$$

$$\Rightarrow x=\frac{y\,logy}{p}-\frac{p}{y}$$

Differentiate w r t y on both sides

$$\Rightarrow \frac{dx}{dy}=\frac{[\,y(\frac{1}{y})+\log y)]p-ylog\,y\frac{dp}{dy}}{p^2}-\frac{y\frac{dp}{dy}-p}{y^2}$$

$$\Rightarrow \frac{1}{p}=\frac{(1+\log y)}{p}-\frac{ylog\,y\frac{dp}{dy}}{p^2}-\frac{1}{y}\frac{dp}{dy}+\frac{p}{y^2}$$

$$\Rightarrow \frac{dp}{dy}\left(\frac{y\,log\,y}{p^2}+\frac{1}{y}\right)-[\frac{p}{y^2}+\frac{(\log y+1)}{p}-\frac{1}{p})=0$$

$$\Rightarrow \frac{dp}{dy}\left[\frac{y^2\log y+p^2}{yp^2}\right]-[\frac{p^2+y^2\log y}{y^2p}]=0$$

$$\Rightarrow \frac{dp}{dy}\frac{1}{yp^2}-\frac{1}{y^2p}=0$$

$$\Rightarrow \frac{dp}{dy}\frac{1}{p}-\frac{1}{y}=0\Rightarrow \frac{dp}{dy}\frac{1}{p}=\frac{1}{y}$$

$$\Rightarrow \frac{dp}{p}=\frac{dy}{y}+\log c$$

$\Rightarrow \log p=\log y+\log c=\log cy$

$\Rightarrow p=cy$ substitute in $y^2\log y = xyp+p^2$

We get log y = cx $+c^2$

2.Solve $p^2y+2px-y=0$

Solution: From the given equation $2x=\frac{y-p^2y}{p}=\frac{y}{p}-py$

$2x=\frac{y}{p}-py$ Differentiate wrt y on both sides

$$2\frac{dx}{dy} = \frac{p - y\frac{dp}{dy}}{p^2} - \left(p + y\frac{dp}{dy}\right) = 0$$

$$\frac{2}{p} = \frac{1}{p} - \frac{y}{p^2}\frac{dp}{dy} - p - y\frac{dp}{dy}$$

$$-\frac{dp}{dy}\left(\frac{y}{p^2} + y\right) - \left(p - \frac{1}{p} + \frac{2}{p}\right) = 0$$

$$\frac{dp}{dy}y\left(\frac{1}{p^2} + 1\right) + \left(p + \frac{1}{p}\right) = 0$$

$$\frac{dp}{dy}y\left(\frac{1}{p^2} + 1\right) + p\left(1 + \frac{1}{p^2}\right) = 0$$

$$\left(1 + \frac{1}{p^2}\right)\left[y\frac{dp}{dy} + p\right) = 0$$

$$consider\ y\frac{dp}{dy} + p = 0 \Rightarrow p = \frac{c}{y}$$

Substitute in $p^2y + 2px - y = 0$ we get $y^2 - 2cx - c^2 = 0$

CLAIRAUT'S EQUATIONS

A differential equation is the in the form $y = px + \varphi(p)$ is called Clairaut's differential equation. The general solution can be determined by $y = cx + \varphi(c)$

Problems: 1 Solve $(y - px)(p - 1) = p$

Solution: The given differential equation can be written as

$y - px = \frac{p}{p-1}$ and is Clairaut's differential equation replace p by c

We get $y - cx = \frac{c}{c-1}$ is the general solution

Problem:2 Solve $y^2 - 2pxy + p^2(x^2 - 1) = m^2$

Solution: The given differential equation can be written as

$$y^2 - 2pxy + p^2x^2 - p^2 = m^2$$

$$y^2 - 2pxy + p^2x^2 = p^2 + m^2$$

$$(y - px)^2 = p^2 + m^2$$

$$\Rightarrow y - px = \pm\sqrt{p^2 + m^2}$$

$\Rightarrow y = px \pm \sqrt{p^2 + m^2}$ and is Clairaut's differential equation replace p by c

$\Rightarrow y = cx \pm \sqrt{c^2 + m^2}$ is the general solution

Problem: 3 Solve $x^2(y - px) = yp^2$ $taking\ x^2 = X\ and\ y^2$=Y

Solution: Let $x^2 = X\ and\ y^2$=Y

$$\Rightarrow 2xdx = dX \quad and\ 2ydy = dY$$

$$\frac{2ydy}{2xdx} = \frac{dY}{dX} = p_1 \Rightarrow \frac{dy}{dx} = \frac{xp_1}{y}$$

$\therefore p = \frac{xp_1}{y}$ put the value in the given DE $x^2(y - px) = yp^2$

$$x^2\left(y - x[\frac{xp_1}{y}]\right) = y[\frac{xp_1}{y}]^2$$

$\Rightarrow x^2[y^2 - x^2p_1] = x^2{p_1}^2$

But $x^2 = X$ and y^2= Y

$\Rightarrow X[Y - Xp_1] = X{p_1}^2$

$\Rightarrow [Y - Xp_1] = {p_1}^2$

$\Rightarrow Y = p_1X + {p_1}^2$ is Clairaut's differential equation replace $p_1$by c

$\Rightarrow Y = cX + c^2$

$\Rightarrow y^2 = cx^2 + c^2$

Problem: 4 Solve $(px - y)(py + x) = 2p$ $taking$ $x^2 = X$ and y^2=Y

Solution: Let $x^2 = X$ and y^2=Y

$\Rightarrow 2xdx = dX \quad and\ 2ydy = dY$

$$\frac{2ydy}{2xdx} = \frac{dY}{dX} = p_1 \Rightarrow \frac{dy}{dx} = \frac{xp_1}{y}$$

$\therefore p = \frac{xp_1}{y}$ put the value in the given DE

$(px - y)(py + x) = 2p$

$\Rightarrow ([\frac{xp_1}{y}]x - y)([\frac{xp_1}{y}]y + x) = 2[\frac{xp_1}{y}]$

$\Rightarrow [x^2p_1 - y^2][xp_1 + x] = 2xp_1$

$\Rightarrow [x^2p_1 - y^2][p_1 + 1] = 2p_1$

But $x^2 = X$ and y^2= Y

$\Rightarrow [Xp_1 - Y][p_1 + 1] = p_1$

$\Rightarrow [Xp_1 - Y] = \frac{p_1}{p_1+1}$

$\Rightarrow Y = p_1X + \frac{p_1}{p_1+1}$ is Clairaut's differential equation replace $p_1$by c

$\Rightarrow Y = cX + \frac{c}{c+1}$

$\Rightarrow y^2 = cx^2 + \frac{c}{c+1}$

&&&&&&&

GOVERNMENT DEGREE COLLEGE, RAVULAPALEM

NAAC Accredited with 'B' Grade(2.61 CGPA)

(Affiliated to Adikavi Nannaya University)

Beside NH-16, Main Road, Ravulapalem-533238, Dr.B.R.Ambedkar Dist., A.P, INDIA

E-Mail : jkcjyec.ravulapalem@gmail.com, Phone : 08855-257061

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UNIT-III, IV & V

HIGHER ORDER LINEAR DIFFERENTIAL EQUATIONS

B. Srinivasa Rao, Lecturer in Mathematics. GDC RVPM

Definition: *A differential equation is in the form*

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{((n-1)} y}{dx^{(n-1)}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \dots\dots.. + a_{(n-1)} \frac{dy}{dx} + a_n = V(x)$$

where $a_0, a_1, a_{2,} \dots . a_{(n-1)}, a_n$ *are constants is called Higher Order Linear Differential Equations of order n with constant coefficients.*

The above differentials equation having two solutions

1. **Complementary Function (C F or y_c)**
2. **Particular Integral (P I or y_p) and the General solution is $y = CF + PI$**

For Example, to find the solution of a differential Equation

$$a \frac{d^2y}{dx^2} + b\frac{dy}{dx} + c\,y = V(x) \text{ where } a, b, c \text{ are constants}$$

1.Complementary Function (CF or y_c):

The CF can be determined from $\mathbf{a \frac{d^2y}{dx^2} + b\frac{dy}{dx} + c\,y = 0}$

$$ie\ (aD^2 + bD + cy) = 0 \quad \text{where } D = \frac{d}{dx} \text{ -------(1)}$$

Suppose $\boldsymbol{y = e^{mx}}$ *is a part of the complementary function.*

$\frac{dy}{dx} = me^{mx}$ and $\frac{d^2y}{dx^2} = m^2 e^{mx}$

Put the values in (1) we get $m^2 e^{mx} + me^{mx} + e^{mx} = 0$

$$e^{mx}(a\,m^2 + b\,m + c) = 0$$

$as\ e^{mx} \neq 0\ for\ any\ m$

we have a m^2+b m+c =0 is a quadratic Equation in m called Auxiliary Equation (AE),It has three types of Roots

i)Real and unequal. ii) Real and equal iii) Complex Roots.

i)If the roots are Real and unequal and they are m_1 &m_2

$$The\ CF = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

ii)If the roots are Real and equal and they are $m_1 = m_2$

$$The\ C\ F = c_1 e^{m_1 x} + c_2 x e^{m_1 x} = [c_1 + x c_2]\, e^{m_1 x}$$

iii)If the roots are Complex and they are in the form m = α ± iβ

$$The\ C\ F = e^{\alpha x}[\, c_1 Cos\ \beta x + c_2 Sin\ \beta x]$$

2. Particular Integral (P I or y_p)

They are five types of functions to finding Particular Integrals

1)e^{ax} 2) sin x or cos x. 3) Polynomial function. 4) $e^{ax}\, V$ 5) x V

Model-1 To find the P I of f(D)y = e^{ax} by

P I = $\frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}$ where f(a) ≠ 0

P I = $\frac{1}{f(D)} e^{ax} = \frac{1}{(D-a)^k} e^{ax} = \frac{x^k}{k\,!} e^{ax}$ where f(a) = 0

Problems on Complementary functions

1.Solve (D^3 + 2D^2 – D – 2) y = 0

Solution: The Auxiliary Equation (AE) is

$$m^3+2m^2 - m - 2 = 0$$

$$m^2(m + 2) - (m + 2) = 0$$

(m^2 -1) (m+ 2) = 0 ⇒ m = ± 1, m = -2.

The solution of the DE is y = $c_1 e^{-2x} + c_2 e^{-1x} + c_3 e^{1x}$

2. Solve (D^3 + 6D^2 + 11D + 6) y = 0

Solution: The Auxiliary Equation (AE) is

$$m^3 + 6m^2 + 11m + 6 = 0$$

Consider x = -1

1	6	11	6
0	-1	- 5	-6
1	5	6	0

The equation $m^2 + 5m + 6 = 0 \Rightarrow (m+2)(m+3) = 0 \Rightarrow m = -2, -3$

$\therefore m = -1, -2, -3$. The solution $\mathbf{y} = \boldsymbol{c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{-3x}}$

3.Solve ($D^3 + 1$) y = 0

Solution: The Auxiliary Equation (AE) is

$m^3 + 1 = 0 \Rightarrow (m+1)(m^2 - m + 1) = 0$

$$\Rightarrow m = -1,\ m = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(1)}}{2(1)}$$

$$\Rightarrow m = -1,\ m = \frac{1 \pm \sqrt{-3}}{2} = \frac{1}{2} \pm \frac{i\sqrt{3}}{2} \text{(complex roots)}$$

The general solution

$$y = \boldsymbol{c_1 e^{-x} + e^{\frac{x}{2}} [\, c_2 \cos \frac{\sqrt{3}}{2} x + c_3 \sin \frac{\sqrt{3}}{2} x \,]}$$

4.Solve ($D^3 - 4D^2 + 4D$) y = 0

Solution: The Auxiliary Equation (AE) is $m^3 - 4m^2 + 4m = 0$

$m(m^2 - 4m + 4) = 0 \Rightarrow m = 0,\ m^2 - 4m + 4 = 0$

$$\Rightarrow m = 0,\ (m-2)^2 = 0$$

$$\Rightarrow m = 0,\ m = 2, 2 \text{ (Repeated roots)}$$

The solution $\mathbf{y} = \boldsymbol{c_1 e^{0x} + c_2 e^{2x} + c_3 x e^{2x}}$

5.Solve ($D^4 + 8D^2 + 16$) y = 0

Solution: The Auxiliary Equation (AE) is $m^4 + 8m^2 + 16 = 0$

$(m^2 + 4)^2 = 0 \Rightarrow m^2 + 4 = 0, m^2 + 4 = 0$

$\Rightarrow m^2 = -4, m^2 = -4 \Rightarrow m = \sqrt{-4}, m = \sqrt{-4} \Rightarrow m = \pm 2\, i, \pm 2\, i$

The general solution $\boldsymbol{y = c_1 \cos 2x + c_2 \sin 2x + x(c_3 \cos 2x + c_4 \sin 2x\,)}$

Model-1

1.Find the value of $\frac{1}{D^2-5D+6}e^x$

Solution: $\frac{1}{D^2-5D+6}e^x = \frac{1}{(1)^2-5(1)+6}e^x = \frac{1}{2}e^x$

***2.Find the Particular integral of* (D^2 - 2D + 1) y =*1+* e^{2x}**

Solution:

$$P\,I = \frac{1}{D^2-2D+1}(1+e^{2x}) = \frac{1}{D^2-2D+1}(e^{0x}) + \frac{1}{D^2-2D+1}(e^{2x})$$

$$= \frac{1}{0^2-2(0)+1}(e^{0x}) + \frac{1}{2^2-2(2)+1}(e^{2x})$$

$$= \frac{1}{1}(e^{0x}) + = \frac{1}{1}(e^{2x}) = 1+e^{2x}$$

3.Solve $(D^2+4D+3)\,y = e^{2x}$

Solution: The A E is $m^2+4m+3=0 \Rightarrow (m+1)(m+3)=0$

$\Rightarrow m = -1, m = -3$

The complementary Function (CF) is

$$y_c = c_1e^{-x} + c_2e^{-3x}$$

$$P\,I = \frac{1}{D^2+4D+3}e^{2x} = \frac{1}{2^2+4(2)+3}e^{2x} = \frac{1}{15}e^{2x}$$

The general solution y = C F +P I = $c_1e^{-x} + c_2e^{-3x} + \frac{1}{15}e^{2x}$

4.Solve $(D^3-5D^2+8D-4)\,y = e^{2x}$

Solution: The A E is $m^3-5m^2+8m-4=0$

m = 1	1	-5	8	-4
	0	1	-4	4
	1	-4	4	0

The quadratic equation $(m-1)(m^2-4m+4)=0$

$$\Rightarrow (m-1)(m-2)^2 = 0 \;\Rightarrow m = 1, 2, 2$$

$$\text{C F} = c_1e^{2x} + c_2xe^{2x} + c_3e^{x}$$

$$\text{P I} = \frac{1}{(D^3 - 5D^2 + 8D - 4)}e^{2x}$$

$$= \frac{1}{(D-1)(D-2)^2}e^{2x}$$

$$= \frac{1}{(2-1)(D-2)^2}e^{2x} = \frac{1}{(D-2)^2}e^{2x} = \frac{x^2}{2!}e^{2x}$$

The General solution $y = c_1e^{2x} + c_2xe^{2x} + c_3e^{x} + \frac{x^2}{2!}e^{2x}$

5.Solve $(D^3 + 1)\, y = 3 + 5e^x$

Solution: The Auxiliary Equation (AE) is

$m^3 + 1 = 0 \Rightarrow (m+1)\,(m^2 - m + 1) = 0$

$$\Rightarrow m = -1,\ m = \frac{-(-1)\pm\sqrt{(-1)^2-4(1)(1)}}{2(1)}$$

$$\Rightarrow m = -1,\ m = \frac{1\pm\sqrt{-3}}{2} = \frac{1}{2} \pm \frac{i\sqrt{3}}{2}\text{(complex roots)}$$

$$y_c = c_1e^{-x} + e^{\frac{x}{2}}\left[c_2 \cos\frac{\sqrt{3}}{2}x + c_3 \sin\frac{\sqrt{3}}{2}x\right]$$

$$\text{P I} = \frac{1}{(D^3+1)}\left[3 + 5e^x\right]$$

$$= \frac{1}{(D^3+1)}3e^{0x} + \frac{1}{(D^3+1)}5e^{1x}$$

$$= \frac{1}{(0^3+1)}3e^{0x} + \frac{1}{(1^3+1)}5e^{1x}$$

$$= \frac{1}{(1)}3e^{0x} + \frac{1}{(2)}5e^{1x} = 3 + \frac{5}{2}e^{x}$$

The General solution

$$y = c_1e^{-x} + e^{\frac{x}{2}}\left[c_2 \cos\frac{\sqrt{3}}{2}x + c_3 \sin\frac{\sqrt{3}}{2}x\right] + 3 + \frac{5}{2}e^{x}$$

6.Solve $(D^2-3D+2)y= \cosh x$

Solution: The Auxiliary Equation (AE) is $m^2 - 3m + 2 = 0$

$$(m-1)\,(m-2) = 0 \Rightarrow m = 1, 2$$

$$\text{C F} = c_1e^{x} + c_2e^{2x}$$

$$P\ I = \frac{1}{D^2 - 3D + 2} \text{Cosh } x$$

$$= \frac{1}{D^2 - 3D + 2}\left[\frac{e^x + e^{-x}}{2}\right]$$

$$= \frac{1}{2}\left[\frac{1}{(D-1)(D-2)}e^x + \frac{1}{(D-1)(D-2)}e^{-x}\right]$$

$$= \frac{1}{2}\left[\frac{1}{(D-1)(1-2)}e^x + \frac{1}{(-1-1)(-1-2)}e^{-x}\right]$$

$$= \frac{1}{2}\left[\frac{-1}{(D-1)}e^x + \frac{1}{6}e^{-x}\right]$$

$$= \frac{1}{2}\left[\frac{-x}{1!}e^x + \frac{1}{6}e^{-x}\right]$$

$$\mathbf{P\ I} = \frac{-x}{2}e^x + \frac{1}{12}e^{-x}$$

The general solution y = $c_1e^x + c_2e^{2x} + \frac{-x}{2}e^x + \frac{1}{12}e^{-x}$

7.Solve $(D^3 - 5D^2 + 7D - 3)y = e^{2x} \cosh x$

Solution: The Auxiliary Equation (AE) is $m^3 - 5m^2 + 7m - 3 = 0$

Consider m = 1

1	-5	7	-3
0	1	-4	3
1	-4	3	0

The equation $m^2 - 4m + 3 = 0 \Rightarrow (m-1)(m-3) = 0 \Rightarrow m = 1, 3$

$\therefore\ m = 1, 1, 3$

$$\mathbf{C\ F} = c_1e^{1x} + c_2xe^{1x} + c_3e^{3x}$$

As $e^{2x} \cosh x = e^{2x}\left[\frac{e^x + e^{-x}}{2}\right] = \left[\frac{e^{3x} + e^x}{2}\right]$

$$\mathbf{P\ I} = \frac{1}{D^3 - 5D^2 + 7D - 3} e^{2x} \cosh x$$

$$= \frac{1}{(D-1)^2(D-3)}\left[\frac{e^{3x} + e^x}{2}\right]$$

$$= \frac{1}{2}\left[\frac{1}{(D-1)^2(D-3)}e^{3x} + \frac{1}{(D-1)^2(D-3)}e^x\right]$$

$$= \frac{1}{2}\left[\frac{1}{(3-1)^2(D-3)}e^{3x} + \frac{1}{(D-1)^2(1-3)}e^x\right]$$

$$= \frac{1}{2}\left[\frac{1}{4(D-3)}e^{3x} + \frac{1}{-2(D-1)^2}e^x\right]$$

$$= \frac{1}{2}\left[\frac{x}{4.1!}e^{3x} + \frac{x^2}{-2.2!}e^x\right]$$

$$=\frac{1}{2}\left[\frac{x}{4}e^{3x}-\frac{x^2}{4}e^{x}\right]=\frac{x}{8}e^{3x}-\frac{x^2}{8}e^{x}$$

The general solution y = $c_1e^x+c_2xe^x+c_3e^{3x}+\frac{x}{8}e^{3x}-\frac{x^2}{8}e^x$

$$y_c=[\,c_1\cos 2x+c_2\sin 2x\,]$$

Model-2

To find the P I of f(D) y = $\sin bx$ or $\cos bx$ by

P I = $\frac{1}{f(D^2)}$ sin bx or cos bx = $\frac{1}{f(-b^2)}$ sin bx or cos bx where $f(-b^2)\neq 0$

P I = $\frac{1}{f(D^2)}$ sin bx or cos bx = P I = $\frac{1}{D^2+b^2}\sin bx \text{ or } \cos bx$ where $f(-b^2)=0$

$$\text{P I}=\frac{1}{D^2+b^2}\;\frac{x}{2}\int \sin bx \text{ or } \cos bx\,dx.$$

Problems

1.Find $\frac{1}{D^2+4}$ **sin2x**

Solution: $\frac{1}{D^2+4}\sin 2x=\frac{x}{2}\int \sin 2x\,dx=\frac{x}{2}\left[\frac{-\cos 2x}{2}\right]=\frac{-x\cos 2x}{4}$

2.Find the $\frac{1}{D^2+D+1}$**sinx**

Solution: $\frac{1}{D^2+D+1}\sin x=\frac{1}{(-1^2+D+1)}\sin x$

$$=\frac{1}{D}\sin x=\int \sin x\,dx=-\cos x$$

3.Solve (D^2+3D+2) y = $e^{-2x}+\sin x$

Solution: The Auxiliary Equation (AE) is

$m^2+3m+2=0\Rightarrow(m+1)(m+2)=0$

$\Rightarrow m=-1,-2$

$$\text{C F}=c_1e^{-1x}+c_2e^{-2x}$$

$$\text{P I}=\frac{1}{D^2+3D+2}[e^{-2x}+\sin x]$$

$$=\frac{1}{D^2+3D+2}e^{-2x}+\frac{1}{D^2+3D+2}\sin x$$

$$=\text{P I}_1+\text{P I}_2\ \text{(Say)}$$

$$P\,I_1 = \frac{1}{D^2+3D+2}\, e^{-2x} = \frac{1}{(D+1)(D+2)} e^{-2x}$$

$$= \frac{1}{(-2+1)(D+2)} e^{-2x}$$

$$= \frac{1}{(-1)(D+2)} e^{-2x}$$

$$= \frac{-1}{(D+2)} e^{-2x} = \frac{-x}{1!} e^{-2x}$$

$$P\,I_2 = \frac{1}{D^2+3D+2} \sin x$$

$$= \frac{1}{-1+3D+2} \sin x$$

$$= \frac{1}{3D+1} \sin x \times \frac{3D-1}{3D-1}$$

$$= \frac{3D-1}{9D^2-1} \sin x$$

$$= \frac{3D\sin x - \sin x}{9(-1)-1} = \frac{3\cos x - \sin x}{-10}$$

The General solution

$$\boldsymbol{y = c_1 e^{-x} + c_2 e^{-2x} - x e^{-2x} + \frac{3\cos x - \sin x}{-10}}$$

4.Solve $(D^2 + 4)\, y = e^x + \sin 2x$

Solution: The Auxiliary Equation (AE) is$m^2 + 4 = 0 \Rightarrow m = \pm 2i$

$$P\,I = \frac{1}{(D^2+4)} [\, e^x + \sin 2x \,]$$

$$= \frac{1}{(D^2+4)} e^x + \frac{1}{(D^2+4)} \sin 2x$$

$$= \frac{1}{(1^2+4)} e^x + \frac{x}{2} \int \sin 2x \, dx$$

$$= \frac{1}{5} e^x + \frac{x}{2} \left(-\frac{\cos 2x}{2}\right)$$

$$= \frac{1}{5} e^x - \frac{x\cos 2x}{4}$$

The general solution y = $[c_1 \cos 2x + c_2 \sin 2x] + \frac{1}{5} e^x - \frac{x\cos 2x}{4}$

5. Solve $(D^2 - 4)\, y = 1 + \cos 2x$.

Solution: The Auxiliary Equation (AE) is $m^2 - 4 = 0 \Rightarrow m^2 = 4 \Rightarrow m = \pm 2$

$$\mathbf{C\ F} = c_1e^{-2x} + c_2e^{2x}$$

$$\mathbf{P\ I} = \frac{1}{D^2-4}(1+\cos 2x)$$

$$= \frac{1}{D^2-4}(1) + \frac{1}{D^2-4}\cos 2x$$

$$= \frac{1}{0^2-4}(e^{0x}) + \frac{1}{-2^2-4}\cos 2x$$

$$= \frac{1}{-4} + \frac{1}{-8}\cos 2x = \frac{-1}{4} - \frac{1}{8}\cos 2x$$

The general solution $y = c_1e^{-2x} + c_2e^{2x} - \frac{1}{4} - \frac{1}{8}\cos 2x$

6. Solve $(D^2 - 4D + 3)\, y = \sin 3x \cos 2x$

Solution: The A E is $m^2 - 4m + 3 = 0 \Rightarrow (m-1)(m-3) = 0 \Rightarrow m = 1, m = 3$

The complementary Function (CF) is $y_c = c_1e^{x} + c_2e^{3x}$

As $\sin 3x \cos 2x = \frac{1}{2}[2 \sin 3x \cos 2x]$

$$= \frac{1}{2}[\sin(3x+2x) + \sin(3x-2x)] = \frac{1}{2}[\sin 5x + \sin x]$$

$$\mathbf{P\ I} = \frac{1}{D^2-4D+3}\sin 3x \cos 2x$$

$$= \frac{1}{D^2-4D+3}\frac{1}{2}[\sin 5x + \sin x]$$

$$= \frac{1}{2}\left[\frac{1}{D^2-4D+3}\sin 5x + \frac{1}{D^2-4D+3}\sin x\right]$$

$$= \frac{1}{2}\left[\frac{1}{-5^2-4D+3}\sin 5x + \frac{1}{-1^2-4D+3}\sin x\right]$$

$$= \frac{1}{2}\left[\frac{1}{-22-4D}\sin 5x + \frac{1}{2-4D}\sin x\right]$$

$$= \frac{1}{2}\left[\frac{1}{-2(11+2D)}\sin 5x + \frac{1}{2(1-2D)}\sin x\right]$$

$$= \frac{1}{2}\left[\frac{11-2D}{-2(121-4D^2)}\sin 5x + \frac{1+2D}{2(1-4D^2)}\sin x\right]$$

$$= \frac{1}{2}\left[\frac{11\sin 5x - 2(5\cos 5x)}{-2[121-4(-25)]} + \frac{\sin x + 2\cos x}{2(1+4)}\right]$$

$$= \frac{1}{2}\left[\frac{11\sin 5x - 10\cos 5x)}{-2[221]} + \frac{\sin x + 2\cos x}{10}\right]$$

$$\mathbf{P\ I} = \frac{-1}{884}[11\sin 5x - 10\cos 5x] + \frac{1}{20}[\sin x + 2\cos x]$$

The general solution is

$$y = c_1e^{x} + c_2e^{3x} - \frac{1}{884}[11\sin 5x - 10\cos 5x] + \frac{1}{20}[\sin x + 2\cos x]$$

7.Solve (D^2- 3D + 2) y = cos3x cos2x

Solution: The A E is $m^2 - 3m + 2 = 0 \Rightarrow (m-1)(m-2) = 0$

$\Rightarrow m = 1, m = 2$

The complementary Function (CF) is

$$y_c = c_1e^{x} + c_2e^{2x}$$

As Cos 3x Cos 2x = $\frac{1}{2}$[2 cos 3x cos 2x]

$= \frac{1}{2}[\cos(3x+2x) + \cos(3x-2x)] = \frac{1}{2}[\cos 5x + \cos x]$

$$\text{P I} = \frac{1}{D^2 - 3D + 2}\cos 3x \cos 2x$$

$$= \frac{1}{D^2 - 3D + 2}\frac{1}{2}[\text{Cos } 5x + \text{Cos } x]$$

$$= \frac{1}{2}\left[\frac{1}{D^2 - 3D + 2}\cos 5x + \frac{1}{D^2 - 3D + 2}\cos x\right]$$

$$= \frac{1}{2}\left[\frac{1}{-5^2 - 3D + 2}\cos 5x + \frac{1}{-1^2 - 3D + 2}\cos x\right]$$

$$= \frac{1}{2}\left[\frac{1}{-23 - 3D}\cos 5x + \frac{1}{1 - 3D}\cos x\right]$$

$$= \frac{1}{2}\left[\frac{1}{-(23 + 3D)}\cos 5x + \frac{1}{(1 - 3D)}\cos x\right]$$

$$= \frac{1}{2}\left[\frac{23 - 3D}{-(529 - 9D^2)}\cos 5x + \frac{1+3D}{(1 - 9D^2)}\cos x\right]$$

$$= \frac{1}{2}\left[\frac{23\text{Cos } 5x - 3(-5\text{Sin } 5x)}{-[529 - 9(-25)]} + \frac{\text{Cos } x + 3(-\text{Sin } x)}{(1+9)}\right]$$

$$= \frac{1}{2}\left[\frac{23\cos 5x + 15\cos 5x)}{-[754]} + \frac{\cos x - 3\sin x}{10}\right]$$

$$\textbf{PI} = \frac{-1}{1508}[23\cos 5x + 15\cos 5x] + \frac{1}{20}[\cos x - 3\sin x]$$

The general solution is y = C F + P I

$$y = c_1e^x + c_2e^{2x} - \frac{1}{1508}[23\cos 5x + 15\cos 5x] + \frac{1}{20}[\cos x - 3\sin x]$$

8.Solve ($D^4 + 3D^2 - 4$)y = $\cos^2 x - \cosh x$

Solution:

The A E is $m^4 + 3m^2 - 4 = 0 \quad \Rightarrow m^4 + 4m^2 - m^2 - 4 = 0$

$$\Rightarrow m^2(m^2 + 4) - (m^2 + 4) = 0$$

$$\Rightarrow (m^2 + 4)(m^2 - 1) = 0$$

$$\Rightarrow m^2 = -4 \quad m^2 = 1 \Rightarrow m = \pm 2i, m = \pm 1$$

$$\text{C F} = c_1 \cos 2x + c_2 \sin 2x + c_3e^{-x} + c_4e^x$$

As $\cos 2x = 2\cos^2 x - 1 \Rightarrow 2\cos^2 x = 1 + \cos 2x \Rightarrow \cos^2 x = \frac{1}{2}[1 + \cos 2x]$

$$\textit{and } \cos hx = \frac{e^x + e^{-x}}{2}$$

$$\text{P I} = \frac{1}{D^4 + 3D^2 - 4}[\cos^2 x - \cos hx]$$

$$= \frac{1}{D^4 + 3D^2 - 4}\cos^2 x - \frac{1}{D^4 + 3D^2 - 4}\cos hx$$

$$= \text{P I}_1 - \text{P I}_2$$

Now $\text{P I}_1 = \frac{1}{D^4 + 3D^2 - 4}\cos^2 x = \frac{1}{(D^2 + 4)(D^2 - 1)}\frac{1}{2}[1 + \cos 2x]$

$$= \frac{1}{2}\left[\frac{1}{(D^2 + 4)(D^2 - 1)} 1 + \frac{1}{(D^2 + 4)(D^2 - 1)}\cos 2x\right]$$

$$= \frac{1}{2}\left[\frac{1}{(0 + 4)(0 - 1)}e^{0x} + \frac{1}{(D^2 + 4)(-2^2 - 1)}\cos 2x\right]$$

$$= \frac{1}{2}\left[\frac{1}{(-4)}e^{0x} + \frac{1}{(D^2 + 4)(-5)}\cos 2x\right]$$

$$= \frac{1}{2}\left[\frac{-1}{4} + \frac{-1}{5}\frac{x}{2}\int \cos 2x \, dx\right]$$

$$\text{P I}_1 = \frac{-1}{8} + \frac{-x}{20}\frac{\sin 2x}{2} = \frac{-1}{8} - \frac{x}{40}\sin 2x$$

$$\text{P I}_2 = \frac{1}{D^4 + 3D^2 - 4}\cos hx = \frac{1}{(D^2 + 4)(D^2 - 1)}\frac{e^x + e^{-x}}{2}$$

$$= \frac{1}{2}\left[\frac{1}{(D^2 + 4)(D - 1)(D + 1)}e^x + \frac{1}{(D^2 + 4)(D - 1)(D + 1)}e^{-x}\right]$$

$= \frac{1}{2}[\frac{1}{(1^2+4)(D-1)(1+1)}e^{x} + \frac{1}{((-1)^2+4)(-1-1)(D+1)}e^{-x}]$

$= \frac{1}{2}[\frac{1}{10(D-1)}e^{x} + \frac{1}{-10(D+1)}e^{-x}]$

$= \frac{1}{2}[\frac{x}{10.1!}e^{x} + \frac{x}{-10.1!}e^{-x}]$

P $I_2 = \frac{x}{10}\frac{1}{2}[\ e^{x} - \ e^{-x}]$= $\frac{x}{10}$ ***sin hx***

The general solution

$$\mathbf{y = c_1 \cos 2x + c_2 \sin 2x + c_3 e^{-x} + c_4 e^{x} - \frac{1}{8} - \frac{x}{40}\sin 2x - \frac{x}{10}\sin hx}$$

Model-3

To find the P I of $f(D)y$ = polynomial function.

P I = $\frac{1}{f(D)}$ (polynomial in x) = $[f(D)]^{-1}$(polynomial function) and apply the binomial expansions

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots \ldots\ldots\ldots\ldots$$

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots \ldots\ldots\ldots\ldots$$

$$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \cdots \ldots\ldots\ldots\ldots$$

$$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \cdots \ldots\ldots\ldots\ldots$$

Calculate them.

Problems

1.Find the Particular integral of $(D + 1)^2$ y = x

Solution: PI = $\frac{1}{(D+1)^2}$ (x) = $\frac{1}{(1+D)^2}$ (x)

$= (1 + D)^{-2}(x)$

= (1 - 2 D + 3 D^2 - + ...) x

P I = (x -2)

2.Solve $(D^2 + D + 1)$ y = x^3

Solution: The Auxiliary Equation (AE) is $m^2 + m + 1 = 0$

$$m = \frac{-1 \pm \sqrt{(1)^2 - 4(1)(1)}}{2(1)}$$

$$\Rightarrow m = -1, m = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1}{2} \pm \frac{i\sqrt{3}}{2} \text{(complex roots)}$$

$$\boldsymbol{y_c = e^{-\frac{x}{2}} [\, c_1 \cos \frac{\sqrt{3}}{2}\, x + c_2 \sin \frac{\sqrt{3}}{2}\, x \,]}$$

$$\text{P I} = \frac{1}{D^2 + D + 1} x^3 = (D^2 + D + 1)^{-1} x^3$$

$$= (1 + D + D^2)^{-1} x^3$$

But $(1 + x)^{-1} = 1 - x + x^2 - x^3 + \ldots\ldots\ldots\ldots\ldots$

$$= [\, 1 - (D + D^2) + (D + D^2)^2 + (D + D^2)^3 + \ldots\ldots\ldots\,] x^3$$

$$= x^3 - 3x^2 - 6x + 6x + 2(6) + 6\,] = \boldsymbol{x^3 - 3x^2 + 18}$$

The general solution $\boldsymbol{y = e^{-\frac{x}{2}} [\, c_1 \cos \frac{\sqrt{3}}{2}\, x + c_2 \sin \frac{\sqrt{3}}{2}\, x \,] + x^3 - 3x^2 + 18}$

3.Solve $(D^3 + 2D^2 + D)\, y = x^2 + x$

Solution: The Auxiliary Equation (AE) is $m^3 + 2m^2 + m = 0$

$m(m^2 + 2m + 1) = 0 \Rightarrow m = 0\ (m^2 + 2m + 1) = 0$

$\Rightarrow m = 0, (m + 1)^2 = 0$

$\Rightarrow m = 0, m = -1, -1$

C F $\boldsymbol{= c_1 e^{-1x} + c_2 x e^{-1x} + c_3 e^{0x} = c_1 e^{-x} + c_2 x e^{-x} + c_3}$

$$\textbf{P I} = \frac{1}{D(D^2 + 2D + 1)} (x^2 + x)$$

$$= \frac{1}{D(D^2 + 2D + 1)} (x^2 + x)$$

$$= \frac{1}{D} (1 + D)^{-2} (x^2 + x)$$

$$= \frac{1}{D} (1 - 2D + 3D^2 - \ldots\ldots\ldots)(x^2 + x)$$

$$= \frac{1}{D} [x^2 + x - 2(2x + 1) + 3(2)]$$

$$= \frac{1}{D} [x^2 - 3x + 4]$$

$$= \int x^2 - 3x + 4 \; dx$$

$$= \frac{x^3}{3} - 3\frac{x^2}{2} + 4x = \frac{1}{6} [\, 2x^3 - 9x^2 + 24x]$$

The general solution $\boldsymbol{y = c_1 e^{-x} + c_2 x e^{-x} + c_3 + \frac{1}{6} [\, 2x^3 - 9x^2 + 24x]}$

4.Solve $(D^2 + 3D + 2)y = e^{-x} + x^2 + \text{Cos } x$

Solution: The Auxiliary Equation (AE) is

$$m^2 + 3m + 2 = 0 \Rightarrow (m+1)(m+2) = 0$$

$$\Rightarrow m = -1, -2$$

$$\textbf{C F} = c_1 e^{-1x} + c_2 e^{-2x}$$

$$P\,I = \frac{1}{D^2+3D+2}\left[e^{-x} + x^2 + \text{Cos } x \right]$$

$$= \frac{1}{D^2+3D+2}(e^{-x}) + \frac{1}{D^2+3D+2}(x^2) + \frac{1}{D^2+3D+2}(\text{Cos } x)$$

P I = P I₁ + P I₂ + P I₃

$$P\,I_1 = \frac{1}{D^2+3D+2}(e^{-x}) = \frac{1}{(D+1)(D+2)}(e^{-x})$$

$$= \frac{1}{(D+1)(-1+2)}(e^{-x}) = \frac{1}{(D+1)}(e^{-x})$$

$$= \frac{x}{1!}(e^{-x}) = \frac{xe^{-x}}{1}$$

$$P\,I_2 = \frac{1}{D^2+3D+2}(x^2) = \frac{1}{2\left(1+\frac{D^2+3D}{2}\right)}(x^2)$$

$$= \frac{1}{2}\left[1 + \frac{D^2+3D}{2}\right]^{-1} x^2$$

$$= \frac{1}{2}\left[1 - \frac{D^2+3D}{2} + \left[\frac{D^2+3D}{2}\right]^2 + \cdots \ldots \ldots\right] \quad x^2$$

$$= \frac{1}{2}\left[x^2 - \frac{2+6x}{2} + \frac{9(2)}{4}\right]$$

$$= \frac{1}{8}\left[4x^2 - 12x + 14\right] = \frac{1}{4}\left[2x^2 - 6x + 7\right]$$

$$P\,I_3 = \frac{1}{D^2+3D+2}(\text{Cos } x) = \frac{1}{-1^2+3D+2}(\text{Cos } x)$$

$$= \frac{1}{1+3D}(\text{Cos } x)$$

$$= \frac{1-3D}{1-9D^2}(\text{Cos } x)$$

$$= \frac{Cos\,x - 3(-Sin\,x)}{1-9(-1)}$$

$$\text{P I}_3 = \frac{Cos\, x + 3Sin\, x}{10}$$

$$\therefore \textbf{P I} = \textbf{P I}_1 + \textbf{P I}_2 + \textbf{P I}_3$$

$$= \frac{xe^{-x}}{1} + \frac{1}{4}[\,2x^2 - 6x + 7\,] + \frac{Cos\, x + 3Sin\, x}{10}$$

The general solution

$$\mathbf{y} = c_1e^{-1x} + c_2e^{-2x} + \frac{xe^{-x}}{1} + \frac{1}{4}[\,2x^2 - 6x + 7\,] + \frac{Cos\, x + 3Sin\, x}{10}$$

5.Solve $(D^2 - 4D + 4)\, y = 8(\, x^2 + e^{2x} + \sin 2x\,)$

Solution: The Auxiliary Equation (AE) is $m^2 - 4m + 4 = 0$

$$(\,m - 2)^2 = 0 \Rightarrow m = 2, 2$$

$$\text{C F} = c_1e^{2x} + c_2xe^{2x}$$

$$\text{P I} = \frac{1}{D^2 - 4D + 4}[\,8(\,x^2 + e^{2x} + \sin 2x\,)\,]$$

$$= 8\,[\,\frac{1}{D^2 - 4D + 4}x^2 + \frac{1}{D^2 - 4D + 4}\,e^{2x} + \frac{1}{D^2 - 4D + 4}\sin 2x\,]$$

$$\text{P I} = 8[\,\text{P I}_1 + \text{P I}_2 + \text{P I}_3\,] \text{ ----------(1)}$$

Now $\text{P I}_1 = \frac{1}{D^2 - 4D + 4}x^2 = \frac{1}{(D-2)^2}x^2$

$$= \frac{1}{4(1 - D/2\,)^2}x^2$$

$$= \frac{(1 - \frac{D}{2})^{-2}}{4}x^2$$

$$= \frac{1}{4}[\,1 + 2(D/2) + 3(\frac{D}{2})^2 + \cdots]x^2$$

$$= \frac{1}{4}[\,x^2 + 2x + \frac{3}{4}(2)]$$

$$P\,I_1 = \frac{1}{8}[\,2x^2 + 4x + 3]$$

$$\text{P I}_2 = \frac{1}{D^2 - 4D + 4}e^{2x}$$

$$= \frac{1}{(D-2)^2}e^{2x} = \frac{x^2}{2\,!}e^{2x}$$

$$\text{P I}_3 = \frac{1}{D^2 - 4D + 4}\sin 2x$$

$= \frac{1}{-2^2 - 4D + 4}$ sin 2x

$= \frac{1}{-4D}$ sin 2x $= \frac{-D}{4D^2}$ Sin 2x

$= \frac{-2\cos 2x}{-4} = \frac{1}{2}$ cos 2x

From (1) P I = 8[P I_1 + P I_2 + P I_3]

P I= 8 [$\frac{1}{8}$[$2x^2$ + 4x +3] + $\frac{x^2}{2!}e^{2x}$ + $\frac{1}{2}$cos 2x

P I = [$2x^2$ + 4x +3 + $4x^2 e^{2x}$ + 4cos 2x]

The general solution

$$y = c_1 e^{2x} + c_2 x e^{2x} + 2x^2 + 4x + 3 + 4x^2 e^{2x} + 4\cos 2x$$

Model-4

To find the P I of $f(D)y = e^{ax} V$ by

P I = $\frac{1}{f(D)} e^{ax} V = e^{ax} \frac{1}{f(D+a)} V$ *and Calculate*

Problems

1.Solve $(D^2 - 4D + 4)y = x e^{2x}$

Solution: The Auxiliary Equation (AE) is $m^2 - 4m + 4 = 0$

$$\Rightarrow (m-2)^2 = 0 \Rightarrow m = 2, 2$$

$$C F = c_1 e^{2x} + c_2 x e^{2x}$$

P I = $\frac{1}{D^2 - 4D + 4} x e^{2x} = \frac{1}{(D-2)^2} x e^{2x}$

$= e^{2x} \frac{1}{[(D+2)-2)]^2} x$

$= e^{2x} \frac{1}{D^2} x = e^{2x} \frac{1}{D} \int x\, dx$

$= e^{2x} \frac{1}{D} \frac{x^2}{2} = e^{2x} \int \frac{x^2}{2}\, dx$

$= e^{2x} [\frac{x^3}{6}] = \frac{x^3}{6} e^{2x}$

The general solution y =$c_1 e^{2x} + c_2 x e^{2x} + \frac{x^3}{6} e^{2x}$

2.Solve $(D^2 - 6D + 13)\, y = 8e^{3x} \sin 2x$

Solution: The Auxiliary Equation (AE) is $m^2 - 6m + 13 = 0$

$$m = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{6 \pm \sqrt{36-52}}{2} = \frac{6 \pm \sqrt{-16}}{2} = \frac{6 \pm 4i}{2} = 3 \pm 2i$$

$$\boldsymbol{y_c = e^{3x}\,[\, c_1 \cos 2\,x + c_2 \sin 2x\,]}$$

$$\text{P I} = \frac{1}{D^2 - 6D + 13}(8\, e^{3x} \sin 2x)$$

$$= 8\, e^{3x} \frac{1}{(D+3)^2 - 6(D+3) + 13}(\sin 2x)$$

$$= 8\, e^{3x} \frac{1}{D^2 + 6D + 9 - 6D - 18 + 13}(\sin 2x)$$

$$= 8\, e^{3x} \frac{1}{D^2+4}(\sin 2x) = 8\, e^{3x} \frac{x}{2} \int \sin 2x\, dx$$

$$= 8\, e^{3x} \left[\frac{-x\cos 2x}{4}\right] = \boldsymbol{2\, x\, e^{3x} \cos 2x}$$

The general solution $y = e^{3x}\,[\, c_1 \cos 2\,x + c_2 \sin 2x\,] + 2\,x\,e^{3x}\cos 2x$

3. Solve $(D^2 - 7D + 6)y = e^{2x}(x + 1)$

Solution: The Auxiliary Equation (AE) is $m^2 - 7m + 6 = 0$

$$\Rightarrow (m-1)(m-6) = 0 \Rightarrow m = 1, 6$$

$$\textbf{C F} = c_1 e^{x} + c_2 e^{6x}$$

$$\textbf{P I} = \frac{1}{D^2 - 7D + 6} e^{2x}(x + 1)$$

$$= e^{2x} \frac{1}{(D+2)^2 - 7(D+2) + 6}(x + 1)$$

$$= e^{2x} \frac{1}{D^2 + 4D + 4 - 7D - 14 + 6}(x + 1)$$

$$= e^{2x} \frac{1}{D^2 - 3D - 4}(x + 1)$$

$$= e^{2x} \frac{1}{-4}\left[1 - \frac{D^2 - 3D}{4}\right]^{-1}(x + 1)$$

$$= e^{2x} \frac{1}{-4}\left[1 + \frac{D^2 - 3D}{4} + \ldots\right](x+1)$$

$$= e^{2x} \frac{1}{-4}\left[x + 1 + \frac{-3(1)}{4}\right] = \frac{e^{2x}}{-16}[4x + 1]$$

The general solution is $y = c_1 e^{x} + c_2 e^{6x} - \frac{e^{2x}}{16}[4x + 1]$

Model-5

To find the P I of $f(D)\,y = x\,V$ by

Formula $\frac{1}{f(D)}\,x\,V = x\frac{1}{f(D)}V - \frac{f'[D]}{[f(D)]^2}\,V$

Problems 1. Solve $(D^2 + 4)\,y = x \sin x$

Solution: The Auxiliary Equation (AE) is $m^2 + 4 = 0 \Rightarrow m^2 = -4 \Rightarrow m = \pm 2i$

$$y_c = [\,c_1 \cos 2x + c_2 \sin 2x\,]$$

$$\text{P I} = \frac{1}{(D^2+4)} x \sin x$$

By the Formula $\frac{1}{f(D)}\,x\,V = x\frac{1}{f(D)}V - \frac{f'[D]}{[f(D)]^2}\,V$

$$= x\,\frac{1}{(D^2+4)}\sin x - \frac{2D}{[\,D^2+4]^2}\sin x$$

$$= x\,\frac{1}{-1^2+4}\sin x - \frac{2\cos x}{[\,-1^2+4]^2}$$

$$= x\,\frac{1}{3}\sin x - \frac{2\cos x}{[3]^2} = \frac{x}{3}\sin x - \frac{2\cos x}{9}$$

The general solution $y = c_1 \cos 2x + c_2 \sin 2x + \frac{x}{3}\sin x - \frac{2\cos x}{9}$

2.Solve $(D^2 + 2D + 1)\,y = x \cos x$

Solution: The A E is $m^2 + 2m + 1 = 0 \Rightarrow (m+1)^2 = 0 \Rightarrow m = -1, -1$

$$C\,F = c_1 e^{-x} + c_2 x e^{-x}$$

$$\text{P I} = \frac{1}{D^2+2D+1} x \cos x$$

We know the formula $\frac{1}{f(D)}\,x\,V = x\frac{1}{f(D)}V - \frac{f'[D]}{[f(D)]^2}\,V$

$$\text{P I} = x\frac{1}{D^2+2D+1}\cos x - \frac{2D+2}{[D^2+2D+1]^2}\cos x$$

$$= x\frac{1}{-1+2D+1}\cos x - \frac{-2\sin x + 2\cos x}{[-1+2D+1]^2}$$

$$= x\frac{1}{2D}\cos x - \frac{-2\sin x + 2\cos x}{[\,2D]^2}$$

$$= x\frac{D}{2D^2}\cos x - \frac{-2\sin x + 2\cos x}{4D^2}$$

$$= x\frac{-\sin x}{2(-1)} - \frac{-2\sin x+2\cos x}{4(-1)}$$

$$\mathbf{P\ I} = \frac{x\sin x}{2} - \frac{\sin x - \cos x}{2}$$

The general solution is y = y c + y p = $c_1e^{-x} + c_2xe^{-x} + \frac{x\sin x}{2} - \frac{\sin x - \cos x}{2}$

3.Solve (D² - 2D +1)y = x sin x

Solution: The A E is $m^2 + 2m + 1 = 0 \Rightarrow (m+1)^2 = 0 \Rightarrow m = -1, -1$

$$C\ F = c_1e^{-x} + c_2xe^{-x}$$

P I = $\frac{1}{D^2+2D+1}$ **x sin x**

We know the formula $\frac{1}{f(D)}\ xV = x\frac{1}{f(D)}V - \frac{f'[D]}{[f(D)]^2}\ V$

$$\mathbf{P\ I} = x\frac{1}{D^2+2D+1}\sin x - \frac{2D+2}{[D^2+2D+1]^2}\ \sin x$$

$$= x\frac{1}{-1+2D+1}\sin x - \frac{2\cos x+2\sin x}{[-1+2D+1]^2}$$

$$= x\frac{1}{2D}\sin x - \frac{2\cos x+2\sin x}{[\,2D]^2}$$

$$= x\frac{D}{2D^2}\sin x - \frac{2\cos x+2\sin x}{4D^2}$$

$$= x\frac{\cos x}{2(-1)} - \frac{2\cos x+2\sin x}{4(-1)}$$

$$= -\ \frac{x\cos x}{2} + \frac{\sin x + \cos x}{2}$$

The general solution is y = y c + y p = $c_1e^{-x} + c_2xe^{-x} - \frac{x\cos x}{2} + \frac{\sin x + \cos x}{2}$

4 . Solve (D² - 1) y = x sin x + x^2e^x

Solution : The auxiliary equation is $m^2 - 1 = 0 \Rightarrow m^2 = 1 \Rightarrow m = \pm 1$

The complementary function $y = c_1e^{-x} + c_2\ e^x$

Particular Integral = $\frac{1}{(D^2-1)}[\ x\sin x + x^2e^x\ ]$

$$= \frac{1}{(D^2-1)}\,x\sin x + \frac{1}{(D^2-1)}\,x^2e^x$$

$$= \mathbf{P\ I_1 + P\ I_2}$$

$$\text{Now P I}_1 = \frac{1}{(D^2-1)} \text{ x sin x}$$

By the Formula $\frac{1}{f(D)}\, x\, V = x \frac{1}{f(D)} V - \frac{f'[D]}{[f(D)]^2}\, V$

$$= x\, \frac{1}{(D^2-1)} \text{ sin x} - \frac{2D}{[\,D^2-1]^2} \text{ sin x}$$

$$\text{P I}_1 = x\, \frac{1}{-1^2-1} \text{ sin x} - \frac{2\cos x}{[\,-1^2-1\,]^2}$$

$$= x\, \frac{1}{-2} \text{ sin x} - \frac{2\cos x}{[-2]^2}$$

$$= \frac{x}{-2} \text{ sin x} - \frac{2\cos x}{4} \quad = \frac{-x}{2} \text{ sin x} - \frac{\cos x}{2}$$

and now $\text{P I}_2 = \frac{1}{(D^2-1)}\, x^2 e^x$

$$= e^x\, \frac{1}{(D+1)^2-1}\, x^2$$

$$= e^x\, \frac{1}{(D^2+2D+1-1)}\, x^2$$

$$= e^x\, \frac{1}{(D^2+2D)}\, x^2$$

$$= e^x\, \frac{1}{2D[\,1+\frac{D}{2}\,]}\, x^2 = e^x \frac{1}{2D}\, [1+\frac{D}{2}\,]^{-1} x^2$$

$$= e^x \frac{1}{2D}\, [1-\frac{D}{2}+(\frac{D}{2})^2 - + \ldots..]\, x^2$$

$$= e^x \frac{1}{2D}\, [\,x^2 - \frac{2x}{2} + \frac{2}{4} \quad = e^x \frac{1}{2D}\, [\,x^2 - x + \frac{1}{2}]$$

$$= e^x \frac{1}{2} \int [\,x^2 - x + \frac{1}{2}\,]dx \quad = e^x\, \frac{1}{2} [\, \frac{x^3}{3} - \frac{x^2}{2} + \frac{x}{2}\,]$$

$$= \frac{e^x}{12} [\,2x^3 - 3x^2 + 3x\,]$$

The general solution

$$y = c_1 e^{-x} + c_2\, e^x - \frac{x}{2} \text{ sin x} - \frac{\cos x}{2} + \frac{e^x}{12} [\,2x^3 - 3x^2 + 3x\,]$$

Unit-IV

Variation of parameters.

To solve a differential equation

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Q\,y = R$$

Where P,Q and R are functions of x then

First to find the complementary function from

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Q\,y = 0$$

and it is $y_c = c_1u(x) + c_2v(x)$

and by the method of variation of parameters the particular integral is in the form

$$y_p = A(x)u(x) + B(x)v(x)$$

Were

$$A(x) = \int \frac{v\,R(x)dx}{vu' - uv'} B(x) = -\int \frac{u\,R(x)dx}{vu' - uv'}$$

Problems 1. Solve $\frac{d^2y}{dx^2} + y = cosec\, x$ *using the method of variation of parameters*.

Solution: The auxiliary equation is $m^2 + 1 = 0 \Rightarrow m = \pm i$

$y_c = c_1\cos x + c_2\sin x$

By the method of variation of parameters

The particular integral $y_p = A(x)u(x) + B(x)v(x)$

Where $A(x) = \int \frac{v\,R(x)dx}{vu' - uv'}$ *and* $B(x) = -\int \frac{u\,R(x)dx}{vu' - uv'}$

Let u = cos x   v = sin x   ⇒   u' = - sin x and v' = cos x

Now $vu' - uv'$ = sin x(- sin x) -cos x (cos x)

$$= -sin^2x - cos^2x = -(sin^2x + cos^2x) = -1$$

∴ $A(x) = \int \frac{v\,R(x)dx}{vu' - uv'} = \int \frac{\sin x(cosc\,x)dx}{-1} = \int(-1)dx =$ - x

$B(x) = -\int \frac{u\,R(x)dx}{vu' - uv'} = -\int \frac{\cos x(cosc\ x)}{-1}\ dx = \int \cot x\ dx = \log|\sin x|$

The particular integral $y_p = A(x)u(x) + B(x)v(x) =$ - x cos x +sin x log | sin x |

The general solution y $= c_1\cos x + c_2\sin x$ - x cos x +sin x log | sin x |

2.Solve $\frac{d^2y}{dx^2} + a^2y = tan\ ax\ using\ the\ method\ of\ variation\ of\ parameters.$

Solution: The auxiliary equation is $m^2 + a^2 = 0 \Rightarrow m = \pm a\,i$

$$y_c = c_1\cos a\,x + c_2\sin ax$$

By the method of variation of parameters

The particular integral $y_p = A(x)u(x) + B(x)v(x)$

Where $A(x) = \int \frac{v\,R(x)dx}{vu' - uv'}$ $and\ B(x) = -\int \frac{u\,R(x)dx}{vu' - uv'}$

Let u = cos ax v = sin ax ⇒ u' = - a sin ax and v' = a cos ax

Now $vu' - uv'$ = sin ax (- a sin ax) -cos ax (a cos ax)

$$= -asin^2ax - a\,cos^2ax = -a\,(sin^2ax + cos^2ax) = -a$$

$$\therefore \quad A(x) = \int \frac{v\,R(x)dx}{vu' - uv'} = \int \frac{\sin ax(\tan ax)dx}{-a}$$

$$= \frac{-1}{a}\int \sin ax\,(\frac{\sin ax}{\cos ax})dx$$

$$= \frac{-1}{a}\int sin^2\,ax\,(\frac{1}{\cos ax})\,dx$$

$$= \frac{-1}{a}\int \frac{1 - cos^2ax}{\cos ax}\,dx$$

$$= \frac{-1}{a}\int \frac{1}{\cos ax} - \frac{cos^2ax}{\cos ax}\,dx$$

$$= \frac{-1}{a}\int [\sec ax - \cos ax]\,dx$$

$$= \frac{-1}{a}\,[\frac{1}{a}\log(\sec ax + \tan ax) - \frac{1}{a}\sin ax]$$

$$= \frac{-1}{a^2}[\log(\sec ax + \tan ax) - \sin ax]$$

$B(x) = -\int \frac{u\,R(x)dx}{vu' - uv'}$ = -$\int \frac{\cos a\,x(\tan ax)}{-a}\, dx$

$$= \frac{-1}{a}\int \cos ax\, [\,\frac{\sin ax}{\cos ax}\,]\, dx$$

$$= \frac{-1}{a}\int \sin ax\;\; dx = \frac{-1}{a}\,[\frac{-\cos ax}{a}\,] = \frac{1}{a^2}\cos \text{ax}$$

The particular integral $y_p = A(x)u(x) + B(x)v(x)$

$$= \cos \text{ax}\,\{\frac{-1}{a^2}\,[\log(\sec \text{ax} + \tan \text{ax}) - \sin \text{ax}\,]\,\} + \sin \text{ax}\,\{\frac{1}{a^2}\cos \text{ax}\,\}$$

$$= \frac{-1}{a^2}\cos \text{ax}\log(\sec \text{ax} + \tan \text{ax})$$

The general solution y $= c_1\cos \text{ax} + c_2\sin \text{ax} - \frac{1}{a^2}\cos \text{ax}\log(\sec \text{ax} + \tan \text{ax})$

3.Solve ($D^2 + a^2$) y $=$ sec a x *using the method of variation of parameters.*

Solution: The auxiliary equation is $m^2 + a^2 = 0 \Rightarrow m = \pm a\, i$

$y_c = c_1\cos \text{a x} + c_2\sin \text{ax}$

By the method of variation of parameters

The particular integral $y_p = A(x)u(x) + B(x)v(x)$

Where $A(x) = \int \frac{v\,R(x)dx}{vu' - uv'}$ *and* $B(x) = -\int \frac{u\,R(x)dx}{vu' - uv'}$

Let u = cos ax, v = sin ax, $\Rightarrow$ u' = - a sin ax and v' = a cos ax

Now $vu' - uv'$ = sin ax(- a sin ax) - cos ax (a cos ax) =

$$-a\sin^2 ax - a\cos^2 ax = -a\,(\sin^2 ax + \cos^2 ax\,) = -a$$

$\therefore \quad A(x) = \int \frac{v\,R(x)dx}{vu' - uv'} = \int \frac{\sin ax(\sec ax)dx}{-a} = \frac{-1}{a}\int \tan ax\, dx$

$$= \frac{-1}{a}\,[\,\frac{1}{a}\log|\sec \text{ax}|\,] = \frac{-1}{a^2}\log|\sec \text{ax}|$$

$B(x) = -\int \frac{u\,R(x)dx}{vu' - uv'}$ = -$\int \frac{\cos a\,x(\sec ax)}{-a}\, dx = \frac{-1}{a}\int 1\, dx = \frac{-x}{a}$

The particular integral $y_p = A(x)u(x) + B(x)v(x)$

$$= \frac{-1}{a^2}\log|\sec \text{ax}|[\cos \text{ax}] - \frac{x}{a}\sin \text{ax}$$

$$= \frac{-\cos ax}{a^2}\log|\sec \text{ax}| - \frac{x}{a}\sin \text{ax}$$

The general solution y = $c_1\cos$ a x + $c_2\sin$ ax $-\frac{\cos ax}{a^2}$ log |sec ax| $-\frac{x}{a}$ sin ax.

4. **Solve $\frac{d^2y}{dx^2} + 4y = 4\,sec^2 2x$ using the method of variation of parameters.**

Solution: The auxiliary equation is $m^2 + 4 = 0 \Rightarrow m = \pm 2\,i$

$$y_c = c_1\cos 2\,x + c_2\sin 2x$$

Let u = cos 2x, v = sin 2x, ⇒ u' = - 2 sin 2x and v' = 2 cos 2x

By the method of variation of parameters

The particular integral $y_p = A(x)u(x) + B(x)v(x)$

Where $A(x) = \int \frac{v\,R(x)dx}{vu' - uv'}$ $and\ B(x) = -\int \frac{u\,R(x)dx}{vu' - uv'}$

Now $vu' - uv'$ = sin 2x (- 2sin 2x) - cos 2x (2 cos 2x) =

$$-2sin^2 2x - 2\,cos^2 2x = -2\,(sin^2 2x + cos^2 2x\,) = -2$$

∴ $\boldsymbol{A(x)} = \int \frac{v\,R(x)dx}{vu' - uv'} = \int \frac{\sin 2x[4\,sec^2 2x\,]\,dx}{-2}$

= -2 $\int \sin 2x\,(\frac{1}{cos^2 2x})$ dx = -2 $\int sec\,2x\,tan\,2x\,dx$

= - sec 2x

$\boldsymbol{B(x)} = -\int \frac{u\,R(x)dx}{vu' - uv'}$ = -$\int \frac{\cos 2x\,(\,4sec^2 2x\,)}{-2}dx$

$= 2\int sec\,2x\,dx$

$= 2[\frac{1}{2}\,log\,(sec\,2x + tan\,2x)]$

$= log\,(sec\,2x + tan\,2x)$

The particular integral $y_p = A(x)u(x) + B(x)v(x)$

= - sec 2x cos 2x + sin 2x $log\,(sec\,2x + tan\,2x)$

= -1 + sin 2x $log\,(sec\,2x + tan\,2x)$

The general solution

y = $c_1\cos 2$ x + $c_2\sin 2$x -1 + sin 2x $log\,(sec\,2x + tan\,2x)$

5.Solve (D^2-3D+ 2) y = sin (e^{-x}) using the method of variation of parameters.

Solution: The auxiliary equation is $m^2 - 3m + 2 = 0 \Rightarrow (m-1)(m-2) = 0$

$$\Rightarrow m = 1, 2$$

$$y_c = c_1 e^x + c_2 e^{2x}$$

By the method of variation of parameters

The particular integral $y_p = A(x)u(x) + B(x)v(x)$

Where $A(x) = \int \frac{v\, R(x)dx}{vu' - uv'}$ and $B(x) = -\int \frac{u\, R(x)dx}{vu' - uv'}$

Let u = e^x v = e^{2x} $\Rightarrow u' = e^x$ and v' =2 e^{2x}

Now u'v – u v' $= e^x(e^{2x}) - e^x(2e^{2x}) = -e^x(e^{2x}) =- e^{3x}$

$$A(x) = \int \frac{v\, R(x)dx}{vu' - uv'} = \int \frac{e^{2x} \sin e^{-x}}{-e^{3x}}\, dx$$

$$= -\int e^{-x} \sin e^{-x}\, dx$$

$$= \int \sin t\, dt \qquad \text{where } t = e^{-x} \Rightarrow dt = -e^{-x}dx$$

$$= -\cos t = -\cos e^{-x}$$

$$B(x) = -\int \frac{u\, R(x)dx}{vu' - uv'} = -\int \frac{e^x \sin e^{-x}}{-e^{3x}}\, dx$$

$$= \int e^{-2x} \sin e^{-x}\, dx$$

$$= \int e^{-x} e^{-x} \sin e^{-x}\, dx$$

$$= -\int t \sin t\, dt \text{where } t = e^{-x} \Rightarrow dt = -e^{-x}dx$$

$$= -[\, t(-\cos t) - (1)(-\sin t)]$$

$$= -[\, -t\cos t + \sin t\,]$$

$$= e^{-x} \cos e^{-x} - \sin e^{-x}$$

The particular integral $y_p = A(x)u(x) + B(x)v(x)$

$$= -\cos e^{-x}[e^x] + [\, e^{-x} \cos e^{-x} - \sin e^{-x}]e^{2x}$$

$$= -e^x \cos e^{-x} + e^x \cos e^{-x} - e^{2x} \sin e^{-x}$$

$$= - e^{2x} \sin e^{-x}$$

The general solution y = $c_1e^x + c_2e^{2x} - e^{2x}sin\ e^{-x}$

6. Solve (D²- 1) y = e⁻ˣ sin(e⁻ˣ) + cos (e⁻ˣ) using the method of variation of parameters.

Solution: The auxiliary equation is $m^2 - 1 = 0 \Rightarrow m = \pm 1$

$$y_c = c_1e^x + c_2e^{-x}$$

By the method of variation of parameters

The particular integral y_p = A(x)u(x) + B(x)v(x)

Where $A(x) = \int \frac{v\,R(x)dx}{vu' - uv'}$ **and** $B(x) = -\int \frac{u\,R(x)dx}{vu' - uv'}$

Let u = e^x v = $e^{-x} \Rightarrow u' = e^x$ and v' = -e^{-x}

Now u' v – u v' $= e^x(e^{-x}) - e^x(-e^{-x}) = 2$

$\therefore \quad A(x) = \int \frac{v\,R(x)dx}{vu' - uv'} = \int \frac{e^{-x}[e^{-x}sin(e^{-x}) + cos\,(e^{-x})]dx}{2}$

$$= \frac{1}{2}\int e^{-x}[e^{-x}sin(e^{-x}) + cos\,(e^{-x})]dx$$

$$= \frac{1}{2}\int [\,t\sin t + \cos t](-dt) \text{where } t = e^{-x} \Rightarrow dt = -e^{-x}dx$$

$$= -\frac{1}{2}[t(-\cos t) - (1)(-\sin t) + \sin t]$$

$$= -\frac{1}{2}[-t\cos t + 2\sin t]$$

$$= \frac{1}{2}[\,e^{-x}\cos e^{-x} - 2\sin e^{-x}\,]$$

$B(x) = -\int \frac{u\,R(x)dx}{vu' - uv'} = -\int \frac{e^x[e^{-x}sin(e^{-x}) + cos\,(e^{-x})]dx}{2}$

$$= \frac{-1}{2}\int [\,sin\,e^{-x} + e^x\,cos\,(e^{-x})]\,dx$$

$$= \frac{-1}{2}\int sin\,e^{-x}dx - \frac{1}{2}\int e^x cos(e^{-x})\,dx$$

$$= \frac{-1}{2}\int sin\,e^{-x}dx - \frac{1}{2}[\,cos(e^{-x})e^x - \int [e^{-x}sin(e^{-x})]\,e^x dx\,]$$

$$= \frac{-1}{2}\int sin\,e^{-x}dx - \frac{1}{2}[e^x\,cos(e^{-x}) - \int sin(e^{-x})\,dx]\,]$$

$$= \frac{-1}{2}\,[\int sin\,e^{-x}dx + e^x\,cos(e^{-x}) - \int sin(e^{-x})\,dx$$

$$= \frac{-1}{2} e^x \cos(e^{-x})$$

The particular integral $y_p = A(x)u(x) + B(x)v(x)$

$$= \frac{e^x}{2}[\, e^{-x} \cos\ e^{-x} - 2 \sin\ e^{-x}\,] + e^{-x}[\frac{-1}{2} e^x \cos(e^{-x})\,]$$

$$= \frac{1}{2}[\cos\ e^{-x} - 2\, e^x \sin\ e^{-x} - \cos e^{-x}\,]$$

$$= -\ e^x \sin\ e^{-x}$$

The general solution

$$y = c_1 e^x + c_2 e^{-x} - e^x \sin\ e^{-x}$$

UNIT-V

CAUCHY-EULER DIFFERENTIAL EQUATIONS

Definition: A differential equation is in the form

$$x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{((n-1)}y}{dx^{(n-1)}} + a_2 x^{n-2} \frac{d^{n-2}y}{dx^{n-2}} + \ldots \ldots .. + a_{n-1} x \frac{dy}{dx} + a_n y = Q(x)$$

where $a_0, a_1, a_{2,} \ldots . a_{(n-1)}, a_n$ *are constants is called Cauchy-Euler Differential Equations of order n.*

To solve the differential equation

Put $x = e^z \Rightarrow z = \log x$

Differentiate w r t x we get $\frac{dz}{dx} = \frac{1}{x}$

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{dy}{dz}\left(\frac{1}{x}\right) \Rightarrow x\frac{dy}{dx} = \frac{dy}{dz} = \theta y \quad where\ \theta = \frac{d}{dz}$$

Similarly, $x^2 \frac{d^2y}{dx^2} = \theta(\theta - 1)y,\ \ x^3 \frac{d^3y}{dx^3} = \theta(\theta-1)(\theta-2)y\ \ etc$

Put these values in the differential equation then we get a linear differential equation with constant coefficients and solve ....

5 Marks Problems:

1.Solve (x^2D^2 - 2 x D – 4) y = x^2

Solution: Given that (x^2D^2 - 2 x D – 4) y = x^2 -----------(1)

Put x = e^z ⇒ z = log x and then

x Dy = $x\frac{dy}{dx} = \theta y$ x^2D^2 y= $x^2\frac{d^2y}{dx^2} = \theta(\theta-1)y$ $where\ \theta = \frac{d}{dz}$

(1) $\Rightarrow \theta(\theta-1)y - 2\,\theta y - 4y = e^{2z}$

$[\theta(\theta-1) - 2\,\theta - 4]y = e^{2z}$

$\Rightarrow [\theta^2 - 3\,\theta - 4]y = e^{2z}$ *is* a linear differential equationwith constant coefficients.

The Auxiliary Equation is $m^2 - 3\,m - 4 = 0$

$m^2 - 4m + m - 4 = 0$

m (m – 4) +(m-4) = 0

(m - 4) (m+1) = 0⇒ m = -1, 4

$y_c = c_1e^{-z} + c_2e^{4z} = c_1x^{-1} + c_2x^4$

$y_p = \frac{1}{\theta^2-3\,\theta-4}e^{2z} = \frac{1}{2^2-3\,(2)-4}e^{2z} = \frac{1}{-6}e^{2z} = \frac{1}{-6}x^2 = \frac{-x^2}{6}$

$The\ general\ solution\ \ y = c_1x^{-1} + c_2x^4 - \frac{x^2}{6}$

2.Solve (x^2 D^2 - 4 x D + 6) y = x^2

Solution: Given that (x^2 D^2 - 4 x D + 6) y = x^2 -----------(1)

Put x = e^z ⇒ z = log x and then $x\frac{dy}{dx} = \theta y$

$x^2\frac{d^2y}{dx^2} = \theta(\theta-1)y$ $where\ \theta = \frac{d}{dz}$

(1) $\Rightarrow \theta(\theta-1)y - 4\,\theta y + 6y = e^{2z}$

$\Rightarrow [\theta(\theta-1) - 4\,\theta + 6]y = e^{2z}$

$\Rightarrow [\theta^2 - 5\,\theta + 6]y = e^{2z}$ *is* a linear differential equationwith constant coefficients.

The Auxiliary Equation is $m^2 - 5\,m + 6 = 0$

$$m^2 - 3m - 2m - 6 = 0$$

$\Rightarrow$ m (m – 3) -2(m-3) = 0

$\Rightarrow$(m - 2)(m - 3) = 0 $\Rightarrow$ m = 2,3

$$y_c = c_1 e^{2z} + c_2 e^{3z} = c_1 x^2 + c_2 x^3$$

$$y_p = \frac{1}{\theta^2 - 5\,\theta - 6} e^{2z} = \frac{1}{(\,\theta-2)(\theta-3)} e^{2z}$$

$$= \frac{1}{(\theta-2)(2-3)} e^{2z} = \frac{-1}{(\theta-2)} e^{2z}$$

$$= \frac{-z}{1\,!} e^{2z} = -x^2 \log x$$

The general solution y = $c_1 x^2 + c_2 x^3 - x^2 \log x$

3. Solve (3 $x^2 D^2$ + x D +1) y = x

Solution: Given that $3x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = x$

Put x = e^z $\Rightarrow$ z = log x and then $x\frac{dy}{dx} = \theta y$

$$x^2 \frac{d^2y}{dx^2} = \theta(\theta - 1)y \quad where\ \theta = \frac{d}{dz}$$

The give D.E can be written as $[3\theta(\theta-1) + \theta + 1]y = e^z$

$\Rightarrow [3\theta^2 - 2\,\theta + 1]y = e^z$ *is* a linear differential equation with constant coefficients.

The Auxiliary Equation is $3m^2 - 2\,m + 1 = 0$

$$m = \frac{2 \pm \sqrt{(2)^2 - 4(1)(3)}}{2(3)} = \frac{2 \pm \sqrt{4-12}}{6} = \frac{2 \pm \sqrt{-8}}{6} = \frac{2 \pm 2i\sqrt{2}}{6} = \frac{1}{3} \pm i\frac{\sqrt{2}}{3}$$

$$\therefore\ m = \frac{1}{3} \pm i\frac{\sqrt{2}}{3}$$

$$y_c = \boldsymbol{e^{\frac{z}{3}}\,[\,c_1 \cos \frac{\sqrt{2}}{3} z + c_2 \sin \frac{\sqrt{2}}{3}\, z\,]}$$

$$y_c = x^{1/3}\left[c_1 \cos\frac{\sqrt{2}}{3}\log x + c_2 \sin\frac{\sqrt{2}}{3}\log x \right]$$

$$y_p = \frac{1}{3\theta^2 - 2\theta + 1} e^z = \frac{1}{3(1) - 2(1) + 1} e^z = \frac{1}{2} e^z = \frac{1}{2}x$$

The general solution $y = x^{1/3}\left[c_1 \cos\frac{\sqrt{2}}{3}\log x + c_2 \sin\frac{\sqrt{2}}{3}\log x \right] + \frac{1}{2}x$

4. Solve $(x^2 D^2 - 3xD + 4)y = 2x^2$

Solution : Given that $x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 4y = 2x^2$

Put $x = e^z \Rightarrow z = \log x$ and then $x\frac{dy}{dx} = \theta y$

$$x^2\frac{d^2y}{dx^2} = \theta(\theta - 1)y \quad where\ \theta = \frac{d}{dz}$$

The give D.E can be written as $[\theta(\theta - 1) - 3\theta + 4]y = 2e^{2z}$

$\Rightarrow [\theta^2 - 4\theta + 4]y = 2e^{2z}$ *is* a linear differential equationwith constant coefficients.

The Auxiliary Equation is $m^2 - 4m + 4 = 0$

$\Rightarrow (m-2)^2 = 0 \Rightarrow m = 2, 2$

$$y_c = c_1 e^{2z} + c_2 z e^{2z} = c_1 x^2 + c_2 x^2 \log x$$

$$y_p = \frac{1}{\theta^2 - 4\theta + 4} 2e^{2z} = \frac{2}{(\theta - 2)^2} e^{2z} = 2\frac{z^2}{2!} e^{2z} = x^2(\log x)^2$$

The general solution $y = c_1 x^2 + c_2 x^2 \log x + x^2(\log x)^2$

5. Solve $(x^2 D^2 + xD - 4)y = x^3$

Solution: Given that $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - 4y = x^3$

Put $x = e^z \Rightarrow z = \log x$ and then $x\frac{dy}{dx} = \theta y$

$$x^2\frac{d^2y}{dx^2} = \theta(\theta - 1)y \quad where\ \theta = \frac{d}{dz}$$

The give D.E can be written as $[\theta(\theta - 1) + \theta - 4]y = e^{3z}$

$\Rightarrow [\theta^2 - 4]y = e^{3z}$ *is* a linear differential equation with constant coefficients.

The Auxiliary Equation is $m^2 - 4 = 0 \Rightarrow m = \pm 2$

$$y_c = c_1 e^{-2z} + c_2 e^{2z} = c_1 x^{-2} + c_2 x^2$$

$$y_p = \frac{1}{\theta^2 - 4} e^{3z} = \frac{1}{3^2 - 4} e^{3z} = \frac{1}{5} x^3$$

The general solution y $= c_1 x^{-2} + c_2 x^2 + \frac{1}{5} x^3$

10 Marks Problems

1. Solve $(x^3D^3+2x^2D^2+2)y=10(x + \frac{1}{x})$

Solution: Given that $x^3 \frac{d^3y}{dx^3} + 2x^2 \frac{d^2y}{dx^2} + 2y = 10(x + \frac{1}{x})$ ---------(1)

Put $x = e^z \Rightarrow z = \log x$ and then $x\frac{dy}{dx} = \theta y$

$$x^2 \frac{d^2y}{dx^2} = \theta(\theta - 1)y$$

$$x^3 \frac{d^3y}{dx^3} = \theta(\theta - 1)(\theta - 2)y where\ \theta = \frac{d}{dz}$$

(1)$\Rightarrow \theta(\theta - 1)(\theta - 2)y + 2\,\theta(\theta - 1)y + 2y = 10(e^z + e^{-z})$

$$\theta(\theta^2 - 3\theta + 2)y + 2(\theta^2 - \theta)y + 2y = 10(e^z + e^{-z})$$

$$(\theta^3 - 3\theta^2 + 2\theta)\,y + (2\theta^2 - 2\theta)y + 2y = 10(e^z + e^{-z})$$

$$(\theta^3 - \theta^2 + 2) = 10(e^z + e^{-z})$$

is a linear differential equation with constant coefficients.

The Auxiliary Equation is $m^3 - m^2 + 2 = 0$

Consider m = -1)

1	-1	0	2
0	-1	2	-2
1	-2	2	0

Now $m^2 - 2\,m + 2 = 0$

$$m = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(2)}}{2(1)} = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i$$

$\therefore$ m = -1, $1 \pm i$

$y_c = c_1 e^{-z} += e^z\,[\,c_1 \cos z + \; c_2 \sin z\,]$

$y_c = c_1 x^{-1}$ +x $[\,c_1 \cos \log x + \; c_2 \sin \log x\,]$

$$y_p = \frac{1}{\theta^3 - \theta^2 + 2} 10(\,e^z + \; e^{-z})$$

$$= 10\left[\,\frac{1}{\theta^3 - \theta^2 + 2} e^z + \frac{1}{\theta^3 - \theta^2 + 2} e^{-z}\,\right]$$

$$= 10\left[\,\frac{1}{(\theta+1)(\theta^2 - 2\,\theta + 2)} e^z + \frac{1}{(\theta+1)(\theta^2 - 2\,\theta + 2)} e^{-z}\,\right]$$

$$= 10\left[\,\frac{1}{(1+1)(1^2 - 2\,(1) + 2)} e^z + \frac{1}{(\theta+1)((-1)^2 - 2\,(-1) + 2)} e^{-z}\,\right]$$

$$= 10\left[\frac{1}{2} e^z + \frac{1}{5(\,\theta+1)} e^{-z}\right] = 10\left[\frac{1}{2} e^z + \frac{z}{5} e^{-z}\right]$$

$$= 5e^z + 2ze^{-z} = 5\,x + 2x^{-1}\log x$$

The general solution

y = $c_1 x^{-1}$ +x $[\,c_1 \cos \log x + \; c_2 \sin \log x\,]$+ 5 x +2 x^{-1} log x

2.Solve (x^2D^2 – x D + 2) y = x log x

Solution: Given $x^2 \frac{d^2y}{dx^2} - x\frac{dy}{dx} + 2y = x \log x$ -----------(1)

Put x = e^z ⇒ z = log x and then $x\frac{dy}{dx} = \theta y$

$x^2 \frac{d^2y}{dx^2} = \theta(\theta - 1)y$ where $\theta = \frac{d}{dz}$

(1)⇒ $\theta(\theta-1)y - \theta y + 2y = ze^z \;\Rightarrow\; (\theta^2 - \theta - \theta + 2)y = ze^z$

$$\Rightarrow\; (\theta^2 - 2\theta + 2)y = ze^z$$

The AE is $m^2 - 2m + 2 = 0$

$$m = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(2)}}{2(1)} = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$$

$$\therefore \ m = 1 \pm i$$

$$y_c = e^z [\, c_1 \cos z + c_2 \sin z \,]$$

$$y_c = x [\, c_1 \cos \log x + c_2 \sin \log x \,]$$

$$y_p = \frac{1}{\theta^2 - 2\theta + 2} z e^z$$

$$= e^z \frac{1}{(\theta+1)^2 - 2(\theta+1) + 2} (z)$$

$$= e^z \frac{1}{\theta^2 + 1} (z)$$

$$= e^z (1 + \theta^2)^{-1} (z) = e^z (1 - \theta^2 + \cdots) z$$

$$= z e^z = x \log x$$

The general solution

$$y = c_1 x^{-1} + x [\, c_1 \cos \log x + c_2 \sin \log x \,] + x \log x$$

3. Solve $(x^2D^2 - xD - 3)y = x^2 \log x$

Solution: Given $x^2 \frac{d^2y}{dx^2} - x \frac{dy}{dx} - 3y = x^2 \log x$ -----------(1)

Put $x = e^z \Rightarrow z = \log x$ and then $x \frac{dy}{dx} = \theta y$

$$x^2 \frac{d^2y}{dx^2} = \theta(\theta - 1)y \quad \text{where } \theta = \frac{d}{dz}$$

$$\theta(\theta - 1)y - \theta y - 3y = z e^{2z}$$

$$\Rightarrow (\theta^2 - \theta - \theta - 3)y = z e^{2z}$$

$$\Rightarrow (\theta^2 - 2\theta - 3)y = z e^{2z}$$

The AE is $m^2 - 2\,m - 3 = 0$

$$(m+1)(m-3) = 0 \Rightarrow m = -1, 3$$

$$y_c = c_1 e^{-z} + c_2 e^{3z} = c_1 x^{-1} + c_2 x^3$$

$$y_p = \frac{1}{\theta^2 - 2\theta - 3} z e^{2z}$$

$$= e^{2z} \frac{1}{(\theta+2)^2 - 2(\theta+2) - 3} (z)$$

$$= e^{2z} \frac{1}{\theta^2 + 2\theta - 3} (z)$$

$$= e^{2z} \frac{1}{-3\left(1 - \frac{\theta^2 + 2\theta}{3}\right)} (z) = \frac{e^{2z}}{-3} [1 - \frac{\theta^2 + 2\theta}{3}]^{-1} (z)$$

$$= \frac{e^{2z}}{-3} [1 + \frac{\theta^2 + 2\theta}{3} + \ \ldots.] \quad (z)$$

$$= \frac{e^{2z}}{-3} [\, z + \frac{2}{3}] = \frac{-x^2}{3} [\frac{2}{3} + \log x\,]$$

The general solution $\quad y = c_1 x^{-1} + c_2 x^3 - \frac{x^2}{3} [\frac{2}{3} + \log x\,]$

4. Solve $(x^2 D^2 - x D + 4) y = \cos(\log x) + x \sin(\log x)$

Solution: Given that $x^2 \frac{d^2 y}{dx^2} - x \frac{dy}{dx} + 4y = \cos(\log x) + x \sin(\log x)$

Put $x = e^z \Rightarrow z = \log x$ and then $x \frac{dy}{dx} = \theta y$

$x^2 \frac{d^2 y}{dx^2} = \theta(\theta - 1) y \qquad$ where $\theta = \frac{d}{dz}$

(1) $\quad \theta(\theta - 1) y - \theta y + 4y = \cos z + e^z \sin z$

$$\Rightarrow \quad (\theta^2 - \theta - \theta + 4) y = \cos z + e^z \sin z$$

$$\Rightarrow \quad (\theta^2 - 2\theta + 4) y = \cos z + e^z \sin z$$

The AE is $m^2 - 2\,m + 4 = 0$

$$m = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(4)}}{2(1)} = \frac{2 \pm \sqrt{4 - 16}}{2} = \frac{2 \pm \sqrt{-12}}{2} = 1 \pm i\sqrt{3}$$

$y_c = e^{z}\,[\,c_1\, cos\, z\sqrt{3} + \; c_2\, sin\, z\sqrt{3}\,]$

y_c= x [$c_1\, cos\, \sqrt{3}log\; x + \; c_2\, sin\sqrt{3}\; log\; x$]

$$y_p \; = \frac{1}{(\theta^2-2\theta+4)} \;\{\, cos\, z + e^{z}\, sin\, z\, \}$$

$$= \frac{1}{(\theta^2-2\theta+4)} cos\, z + \frac{1}{(\theta^2-2\theta+4)} e^{z}\, sin\, z$$

$$= \frac{1}{(-1^2-2\theta+4)} cos\, z + e^{z} \frac{1}{(\theta+1)^2-2(\theta+1)+4)} sin\, z$$

$$= \frac{1}{(3-2\theta)} cos\, z + e^{z} \frac{1}{(\theta^2+2\theta+1\, -2\theta-2+4)} sin\, z$$

$$= \frac{3+2\theta}{(9-4\theta^2)} cos\, z + e^{z}\; \frac{1}{(\theta^2+3)} sin\, z$$

$$= \quad \frac{3\cos z - 2\sin z}{(9+4)} + e^{z} \frac{1}{(-1^2+3)} sin\, z$$

$$= \frac{3}{13}\;(3\cos z - 2\sin z) + \frac{e^{z}}{2}\sin z$$

$$= \frac{3}{13}\;(3\cos\log x - 2\sin\log x) + \frac{x}{2}\sin\log x$$

The general solution

y= x [$c_1\, cos\, \sqrt{3}log\; x + \; c_2\, sin\sqrt{3}\; log\; x$] $+ \frac{3}{13}\;(3\cos\log x - 2\sin\log x) + \frac{x}{2}\sin\log x$

5. Solve $(x^2D^2 + x\,D + 1)\,y = \sin(\log x)$

Solution: Given that $\quad x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = \sin(\log x)$

Put $x = e^z \Rightarrow z = \log x$ and then $x\frac{dy}{dx} = \theta y$

$x^2\frac{d^2y}{dx^2} = \theta(\theta-1)y \qquad$ where $\theta = \frac{d}{dz}$

The given differential equation can be written as

$$\theta(\theta-1)y \;+\; \theta y \;+ y \;=\; \sin z$$

$$[\theta^2 - \theta \;+\; \theta + 1]y = \sin z$$

$$[\theta^2 + 1]y = \sin z$$

The auxiliary equation is $m^2 + 1 = 0 \Rightarrow \ m^2 = -1 \ \Rightarrow m = \ \pm i$

$$y_c = c_1 \ \cos z + c_2 \sin z = c_1 \ \cos \log x + c_2 \sin \log x$$

Now $y_p = \frac{1}{\theta^2+1}$ sin z $= \frac{z}{2} \int \sin z dz = \frac{z}{2}$ (-cos z) = -log x cos(log x)

The general solution y = $c_1 \ \cos \log x + c_2 \sin \log x -$ log x cos(log x)

6. Solve (x^2 D^2 – x D +1) y = log x

Solution : Given that $\quad x^2 \frac{d^2y}{dx^2} - x\frac{dy}{dx} + y =$ log x

Put x = e^z ⇒ z = log x and then $x\frac{dy}{dx} = \theta y$

$$x^2 \frac{d^2y}{dx^2} = \theta(\theta - 1)y \qquad \text{where } \theta = \frac{d}{dz}$$

The given differential equation can be written as

$$\theta(\theta - 1)y \ - \ \theta y \ + y \ = z$$

$$[\theta^2 - \theta - \ \theta + 1]y = z$$

$$[\ \theta^2 - 2\theta + 1 \ ]y = z$$

The auxiliary equation is $m^2 - 2m + \ 1 = 0 \Rightarrow \ (m - 1)^2 = 0 \ \Rightarrow m = 1, 1$

$$y_c = c_1 \, e^z + \ c_1 \, ze^z = c_1 \, x + c_2 \, x \log x$$

$$y_p = \frac{1}{(\theta-1)^2} \ z \ = (1 - \ \theta)^{-2} \, z = (\ 1 + 2\theta + \cdots \ldots \ldots . .)z = z + 2 = 2 + \log x$$

The general solution is y = $c_1 \, x + c_2 \, x \log x + \ 2 + \log x$

7. Solve (x^2 D^2 – 3 x D + 5) y = x^2 sin (log x)

Solution: Given that $\quad x^2 \frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 5y = x^2 \, sinlog \, x$

Put x = e^z ⇒ z = log x and then $x\frac{dy}{dx} = \theta y$

$$x^2 \frac{d^2y}{dx^2} = \theta(\theta - 1)y \qquad \text{where } \theta = \frac{d}{dz}$$

The given differential equation can be written as

$$\theta(\theta-1)y \; - \; 3\theta y \; + 5\mathrm{y} \; = e^{2z} sinz$$

$$[\theta^2 - \theta - \; 3\theta + 5\,]y = e^{2z} sinz$$

$$[\;\theta^2 - 4\theta + 5\,]y = e^{2z} sinz$$

The auxiliary equation is $m^2 - 4m + \; 5 = 0$

$$\mathrm{m} = \frac{4 \pm \sqrt{(4)^2 - 4(1)(5)}}{2(1)} \quad = \frac{4 \pm \sqrt{16 - 20}}{2} = \frac{4 \pm \sqrt{-4}}{2} = 2 \pm i$$

$$\therefore \mathrm{m} = 2 \pm i$$

$$y_c = e^{2z}\,[\,c_1 \cos z + \; c_2 \sin z\,] = x^2[\,c_1 \cos \log x + \; c_2 \sin \log x\,]$$

$$y_p = \frac{1}{\theta^2 - 4\theta + 5}\,(e^{2z} sinz\,)$$

$$= e^{2z}\,\frac{1}{(\theta+2)^2 - 4(\theta+2) + 5}\,(sinz\,)$$

$$= \; e^{2z}\,\frac{1}{\theta^2 + 4\theta + 4 - 4\theta - 8 + 5}\,(sinz\,)$$

$$= \; e^{2z}\,\frac{1}{\theta^2 + 1}\,(sinz\,)$$

$$= \; e^{2z}\;\frac{z}{2}\int \sin z dz$$

$$= e^{2z}\,\frac{z}{2}(\text{- cos z}) = \text{ - }\frac{x^2 \log \mathrm{x}}{2}\;\cos(\log \mathrm{x})$$

The general solution is

$$\mathrm{y} = x^2\left[\,c_1 \cos \log x + \; c_2 \sin \log x - \frac{x^2 \log \mathrm{x}}{2}\,\cos(\log \mathrm{x})\,\right]$$

All the Best